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FINITE ELEMENT METHOD (FEM)

→ Finite element method sometimes referred ~~as~~ to as finite element analysis (FEA) is a ~~com~~ numerical method ~~used to obtain appro~~ like finite difference method but it is more general and powerful in its application to real world problems that involve complicated physics, geometry and/or boundary condition.

→ In finite element given domain (geometric region over which which equations are solved) is discretized as a collection of simple subdomains and over ~~which~~ each subdomain the governing eq^n is approximated by any traditional variational method. Each subdomains are called as finite elements.

~~FEM involves~~ steps involved in ~~FEM~~ FEM

1) Devision of whole domain into subdomains

Discretization (or representation) of given domain into a collection of preselected finite elements.

a) Construct the finite element mesh of preselected elements
b) Number the nodes and elements
c) Generate the geometric properties (eg coordinates and c/s areas) needed for the problem.

2) Derivation of approximation functions over each element

→ The ~~algebraic~~ approximation functions are often algebraic polynomials that are derived using interpolation theory.

3) Assembly of elements

Assembly of element equation to obtain the eq^n of whole problem.

a) Indentify the interelement continuity conditions among the primary variables (relationship between the local degrees of freedom and global degrees of freedom – connectivity of elements) by relating ~~the~~ element nodes to global nodes.

b) Indentify the "equilibrium" conditions among secondary vari
(relationship between force components and the globally specified source components).

c). Assemble element eq^n.

4. Imposition of boundary condition

a) Indentify the specified global primary degrees of freedom (desplacements

b) Indentify the specified global degrees of freedom.

5- Solution of assembled eq^n

6

Advantages of FEA over other numerical analysis

1) ~~The method can efficiently be applied to cater~~

1) applicable to field problem eg Heat transfer, stress analysis magentic field and so on

2) can be applied to any geometrical shape.

3) can handle the all boundary condition and loading

4) All material properties are allowed.

5) can take ~~materials having~~ component that have diff behaviour and deffrent mathematical description

6) An F.E structure closely resembles the actual body region to be analysed.

7) The approximation is easily improved.

In 1941, Hrenikoff

DIFFERENCE between FEM & Finite difference method (FDM)

→ In FDM, field variable is computed at specified points only whereas in FEM field variable is computed at nodes and boundary of the element.

→ FDM models differential eqn while FEM closely model the physical problem at hand.

→ FDM needs larger no. of nodes to get good results while FEM needs fewer nodes.

→ FDM handles complicated problems faerly where as FDM can handle all complicated problems easily

→ FDM makes stair type approximation to sloping and curved boundaries. FEM can consider the sloping boundaries exactly and even curved boundaries can be handled exactly

→ FEM is more general and powerful techniques in real-field problems than FDM.

Similaraties

→ In both cases, differential equations are converted into algebraic eqn.

→ The integration points in the finite difference method are analogus to the nodes in finite element model.

→ If the intregation step (step size) in FDM is reduced, the solution is expected to converge to the exact soln. This is similar to the expected convergence of finite element solution as the mesh of elements is refined.

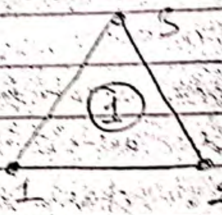
→ In both cases, the refinement represents reduction of mathematical model from finite to infinitesimal.

Nodes, Element and node numbering

Consider ~~a~~ the two dimensional region is divided into sided triangles.

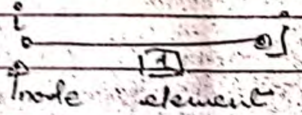
→ The points ~~where~~ where the corners of triangles meet are called as nodes and each triangle formed by three nodes and three sides is called as el

fig :- ~~two~~ finite elemen discretizatio



Consider element ①, 1, 2, 3 is node numbe usually in anticlockwise direction.

Eg line element :



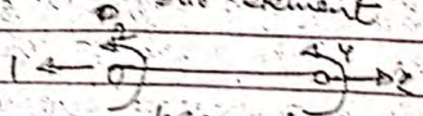
Types of element

1) one dimensional element (line element)

a)

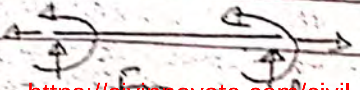
bar element

b)



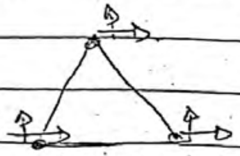
beam element

c)

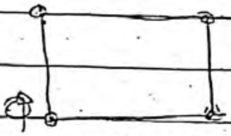


Frame element

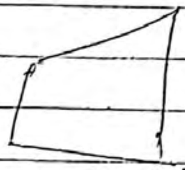
2) 2-D element plane element



triangular element

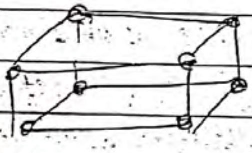


rectangular element



Quadrilateral element

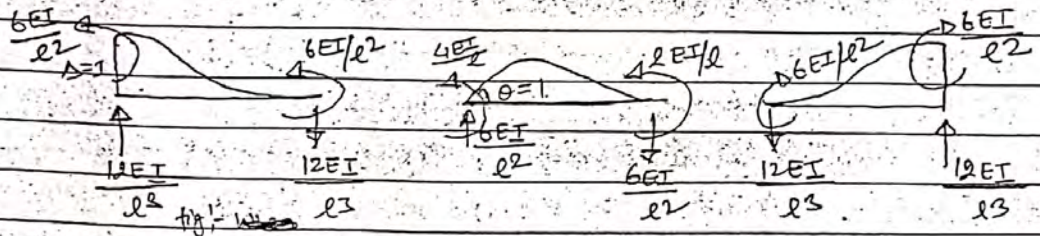
3) 3D Element (solid element)

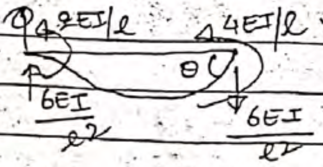


Direct stiffness matrix

For bar element and truss element, it is done in respective chapter

→ For beam element (As done in theory of structure II)





$$[K] = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} = \begin{bmatrix} \frac{12EI}{\ell^3} & \frac{6EI}{\ell^2} & \frac{-12EI}{\ell^3} & 6EI/\ell^2 \\ 6EI/\ell^2 & \frac{4EI}{\ell} & -6EI/\ell^2 & 2EI/\ell \\ -12EI/\ell^3 & \frac{-6EI}{\ell^2} & 12EI/\ell^3 & -6EI/\ell^2 \\ 6EI/\ell^2 & \frac{2EI}{\ell} & -6EI/\ell^2 & \frac{4EI}{\ell} \end{bmatrix}$$

One dimensional bar element

Q No. 1 Consider a thin steel plate as shown in fig. The plate has a uniform thickness $t = 25$ mm, $E = 2*10^5 N/mm^2$ $\gamma = 78 \, KN/m^3$ in addition to the self weight; The plate is subjected to point load $p = 50$ kg at end as shown.

a) Model the plate with two finite element

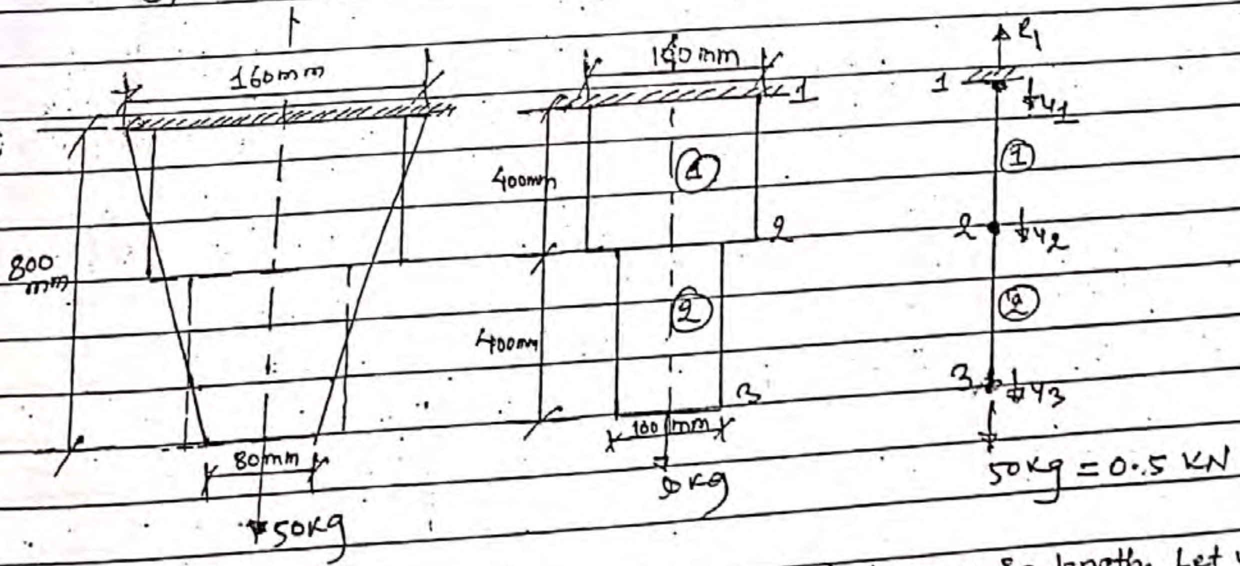
b) Write down the expressions for stiffness matrices and body force vectors.

c) Assemble the structural stiffness matrix $[K]$ and global load vector $\{F\}$

d) Using elimination approach, solve the global displacement vector

e) Evaluate the stresses in each element

f) Determine the reaction force at support.



Sol^n Let us use two elements, each of 400 mm in length. Let us use two bar elements and take average cross section.

width of plate at mid section = $\frac{160+80}{2}$ = 120 mm

average width of element ① = $\frac{120+160}{2}$ = 140 mm

average width of element ② = $\frac{120+80}{2}$ = 100 mm

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Stiffness matrix

For element 1

$$[K] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K^1] = \frac{A_1 E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{3500 * 2 * 10^5}{400}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Global Dof

$$= 10^3 \begin{array}{cc|c} 1 & 2 & \\ \hline 1750 & -1750 & 1 \\ -1750 & 1750 & 2 \end{array}$$

For element 2

Global Dof

$$[K^2] = \frac{2500 * 2 * 10^5}{400}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^3 \begin{array}{cc|c} 2 & 3 & \\ \hline 1250 & -1250 & 2 \\ -1250 & 1250 & 3 \end{array}$$

Now, Assembling $[K^1]$ and $[K^2]$

$$\text{Global stiffness matrix } [K] = 10^3 \begin{array}{ccc|c} 1 & 2 & 3 & \\ \hline 1750 & -1750 & 0 & 1 \\ -1750 & 3000 & -1250 & 2 \\ 0 & -1250 & 1250 & 3 \end{array}$$

Element body force vector.

$$\boxed{\{f\}^e = \frac{\overset{\gamma}{f} A l}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}$$

$$\{f\}^1 = \frac{78\,\text{KN/m}^3 * 3500\,\text{mm}^2 * 400\,\text{mm}}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\{f^1\} = \frac{7.8 * 10^{-5} * 3500 * 400}{2}\begin{Bmatrix}1\\1\end{Bmatrix} = \begin{Bmatrix}54.6\\54.6\end{Bmatrix}\begin{matrix}1\\2\end{matrix}$$

$$\{f^2\} = \frac{7.8 * 10^{-5} * 2500 * 400}{2}\begin{Bmatrix}1\\1\end{Bmatrix} = \begin{Bmatrix}39\\39\end{Bmatrix}\begin{matrix}2\\3\end{matrix}$$

on assembling,

$$\{f\} = \begin{Bmatrix}54.6\\93.6\\39\end{Bmatrix}\begin{matrix}1\\2\\3\end{matrix}$$

externally applied load vector $= \begin{Bmatrix}0\\0\\500\end{Bmatrix}\begin{matrix}1\\2\\3\end{matrix}$ ∵ $50\,kg = 500\,N$

let R_1 be the support reaction at node ①, then

- sign due to the direction of Reaction is opposite to the assumed dirn

Global load vector $\{F\} = \begin{Bmatrix}54.6+0-R_1\\93.6+0\\39+500\end{Bmatrix} = \begin{Bmatrix}54.6-R_1\\93.6\\539\end{Bmatrix}\begin{matrix}1\\2\\3\end{matrix}$

Now Global equilibrium,

$$[K]\{u\} = \{F\}$$

$$10^3\begin{array}{c}\begin{matrix}1 & 2 & 3\end{matrix}\\ \begin{bmatrix}1750 & -1750 & 0\\ -1750 & 3000 & -1250\\ 0 & -1250 & 1250\end{bmatrix}\end{array}\begin{Bmatrix}u_1\\u_2\\u_3\end{Bmatrix} = \begin{Bmatrix}54.6-R_1\\93.6\\539\end{Bmatrix}\begin{matrix}1\\2\\3\end{matrix}$$

Applying boundary condition, $u_1 = 0$, deleting rows and column corresponding to 1, we get following reduced form'

$$10^3 \begin{bmatrix} 3000 & -1250 \\ -1250 & 1250 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 93.6 \\ 539 \end{Bmatrix}$$

on solving, $u_2 = 3.61 * 10^{-4}$ mm

$u_3 = 7.93 * 10^{-4}$ mm

Stresses

$$\sigma = E \cdot \varepsilon$$
$$= E[B]\{\underline{u}\}$$

Now, $[B^1] = \frac{1}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} = \frac{1}{400} \begin{bmatrix} -1 & 1 \end{bmatrix}$

$[B^2] = \frac{1}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} = \frac{1}{400} \begin{bmatrix} -1 & 1 \end{bmatrix}$

Now, $\sigma_1 = 2*10^5 * \frac{1}{400} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{2*10^5}{400} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 3.61*10^{-4} \end{Bmatrix}$

$= 0.181 \text{ N/mm}^2$

$\sigma_2 = 2*10^5 * \frac{1}{400} \begin{bmatrix} -1, 1 \end{bmatrix} \begin{Bmatrix} 3.61*10^{-4} \\ 7.93*10^{-4} \end{Bmatrix} = 0.216 \text{ N/mm}^2$

Reaction Constr

We have, $10^3 \begin{bmatrix} 1750 & -1750 & 0 \\ -1750 & 3000 & -1250 \\ 0 & -1250 & 1250 \end{bmatrix} \begin{Bmatrix} 0 \\ 3.61*10^{-4} \\ 7.93*10^{-4} \end{Bmatrix} = \begin{Bmatrix} 54.6 - R_1 \\ 93.6 \\ 539 \end{Bmatrix}$

Constraint equation.

$$10^3 \left(1750*0 - 1750*3.61*10^{-4} + 0*7.93*10^{-4}\right) = 54.6 - R_1$$

$R_1 = 686.35$ N. Ans

QNo:2. Consider a bar element of 200 mm length. c/s area of the bar is 2500 mm^2 and Young's modulus of elasticity $2*10^5$ N/mm^2. If nodal displacements of the element $u_1 = 0.5$ mm and $u_2 = 0.7$ mm, determine the

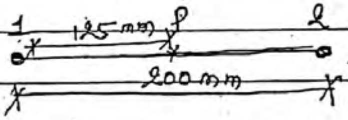
a) displacement at point p at 125 mm from 1st node

b) the strain and stresses

c) the element stiffness matrix

d) the strain energy in the element

Given



length $l = 200mm = 0.2m$

c/s $A = 2500mm^2$

$= 2.5*10^{-3}\ m^2$

Young's modulus $E = 2*10^5 N/mm^2 = 2*10^{11} N/m^2$

nodal displacements $u_1 = 0.5\ mm$ $\quad u_2 = 0.7\ mm$

a) We have to determine the displacements at point p, 125 mm from the 1st node

Where, $\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1 = \frac{2(125-0)}{200} - 1 = 0.25$

Displacement at point p $\quad u(\xi) = N_1 u_1 + N_2 u_2$

Where $N_1 = \frac{1-\xi}{2} = \frac{1-0.25}{2} = 0.375$

$N_2 = \frac{1+\xi}{2} = \frac{1+0.25}{2} = 0.625$

$u(0.25) = 0.0375 * 0.5 + 0.625 * 0.7 = 0.625\ mm$

Ans

b) We have, $\varepsilon = [B]\{\underline{u}\}$

Where $[B] = \frac{1}{x_2 - x_1}\begin{bmatrix} -1 & 1 \end{bmatrix}$

$\{\underline{u}\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

$$[B] = \frac{1}{200}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$u = \begin{Bmatrix} 0.5 \\ 0.7 \end{Bmatrix} mm$$

$$\varepsilon = \frac{1}{200}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} 0.5 \\ 0.7 \end{Bmatrix} = 1*10^{-3} \quad Ans$$

stress $\sigma = E \varepsilon = 2*10^{11} * 1*10^{-3} = 2*10^{8}\ N/m^2$

c) <u>Element stiffness matrix</u>

$$K = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{2500*2*10^{5}}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2.5*10^{6}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

d) Strain energy in the element

$$\boxed{Ve = \frac{1}{2}\{u\}^T[K]\{u\}} = \frac{1}{2}\begin{bmatrix} 0.5 & 0.7 \end{bmatrix} 2.5*10^{6}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0.5 \\ 0.7 \end{Bmatrix}$$

$$= \frac{1}{2}*2.5*10^{6}\begin{bmatrix} -0.2 & 0.2 \end{bmatrix}\begin{Bmatrix} 0.5 \\ 0.7 \end{Bmatrix}$$

$$= 5*10^{4}\ N\text{-}mm \quad Ans$$

QNo.3. Con

nodal d

Sol^n.

elem

The

For e

$[K^1$

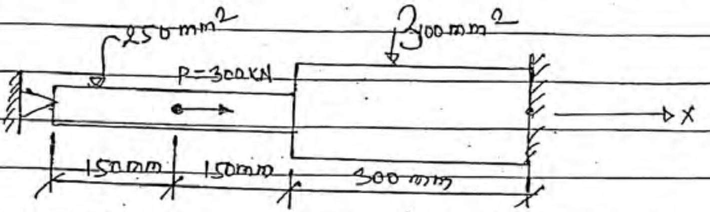
For

$[K^2$

For

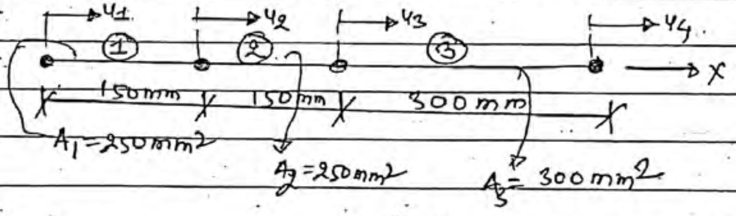
$[K^3] = \frac{E_3}{l}$

QNo.3. Consider the bar as shown in fig. Determine the nodal displacements, element stresses and support reactions.



$$E = 200\times10^{9}\ N/m^2 = 200\times10^{9}\times10^{-6} = 200\times10^{3}\ N/mm^2$$
$$(1\ kN = 1000\ N) \qquad = 2\times10^{5}\ N/mm^2$$

Sol^n 1st modelling of the structure, let us use three elements.



The element stiffness matrices are

For element ①

$$[K^1] = \frac{E_1A_1}{l_1}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = \frac{2\cancel{00}\times10^{\cancel{9}\,5}\times250}{150}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

$$= 10^{5}\begin{array}{c}\begin{array}{cc}1 & 2\end{array}\\ \begin{bmatrix}3.333 & -3.333\\-3.333 & 3.333\end{bmatrix}\end{array}\begin{array}{c}1\\2\end{array}$$

For element ②

$$[K^2] = \frac{E_2A_2}{l_2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = \frac{2\cancel{00}\times10^{\cancel{9}\,5}\times250}{150}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = \cancel{2\times}10^{5}\begin{array}{c}\begin{array}{cc}2 & 3\end{array}\\ \begin{bmatrix}3.333 & -3.333\\-3.333 & 3.333\end{bmatrix}\end{array}\begin{array}{c}2\\3\end{array}$$

For element ③

$$[K^3] = \frac{E_3A_3}{l_3}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = \frac{2\cancel{00}\times10^{\cancel{9}\,5}\times300}{300} = 10^{5\,\cancel{00}}\begin{array}{c}\begin{array}{cc}3 & 4\end{array}\\ \begin{bmatrix}2 & -2\\-2 & 2\end{bmatrix}\end{array}\begin{array}{c}3\\4\end{array}$$

$$\begin{array}{c|c} \begin{bmatrix} -3.333 & 6.666 & -3.333 & 0 \\ 0 & -3.333 & 5.333 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \\ 4 \end{array} \end{array}$$

No body force vector because weights acts ~~ver~~ towards vertically downward. But this is one dimensional element, so no effect of weight towards global x-axis.

$$\text{point load vector } \{P\} = \begin{Bmatrix} 0 \\ 300000 \\ 0 \\ 0 \end{Bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \text{ N}$$

let R_1 and R_4 be the horizontal reaction at ① ④ node 1 and 4, then,

$$\text{Global load vector } \{F\} = \begin{Bmatrix} 0+R_1 \\ 30000 \\ 0 \\ 0+R_4 \end{Bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Now, Global equilibrium, $[K]\{u\} = \{F\}$

$$10^{5} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 3.333 & -3.333 & 0 & 0 \\ -3.333 & 6.666 & -3.333 & 0 \\ 0 & -3.333 & 5.333 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 300000 \\ 0 \\ R_4 \end{Bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Applying boundary condition $u_1 = u_4 = 0$, deleting rows and columns corresponding to 1 and 2, we get follow reduced form.

$$10^5 \begin{bmatrix} 6.666 & -3.333 \\ -3.333 & 5.333 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 300000 \\ 0 \end{Bmatrix}$$

On solving

$u_2 = 0.654$ mm

$u_3 = 0.41$ mm

Now, element stresses

$$\sigma^{e} = {}^{e}E\varepsilon = E[B]\{u\}$$

For element 1.

$$[B] = \frac{1}{x_2 - x_1}[-1 \quad 1] = \frac{1}{150}[-1 \quad 1] \quad \& \quad \{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.654 \end{Bmatrix}$$

$$\therefore \sigma^1 = 2*10^5 * \frac{1}{150}[-1 \quad 1] \begin{Bmatrix} 0 \\ 0.654 \end{Bmatrix} = \frac{2*10^5 * 0.654}{150} = 872 \text{ N/mm}^2$$

For element2

$$[B] = \frac{1}{x_2 - x_1}[-1 \quad 1] = \frac{1}{150}[-1 \quad 1] \quad \& \quad \{u\} = \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0.654 \\ 0.41 \end{Bmatrix}$$

$$\therefore \sigma^2 = 2*10^5 * \frac{1}{150}[-1 \quad 1] \begin{Bmatrix} 0.654 \\ 0.41 \end{Bmatrix} = \frac{2*10^5}{150}(-0.244) = -325.33 \text{ N/mm}^2$$

For element 3

$$[B] = \frac{1}{x_2 - x_1}[-1 \quad 1] = \frac{1}{300}[-1 \quad 1] \quad \& \quad \{u\} = \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0.41 \\ 0 \end{Bmatrix}$$

$$\therefore \sigma^3 = 2*10^5 * \frac{1}{300}[-1 \quad 1] \begin{Bmatrix} 0.41 \\ 0 \end{Bmatrix} = \frac{2*10^5}{300}(-0.41) = -273.33 \text{ N/mm}^2$$

Reaction

$$10^5 \begin{bmatrix} 3.333 & -3.333 & 0 & 0 \\ -3.333 & 6.666 & -3.333 & 0 \\ 0 & -3.333 & 5.333 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.654 \\ 0.41 \\ 0 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 300000 \\ 0 \\ R_4 \end{Bmatrix}$$

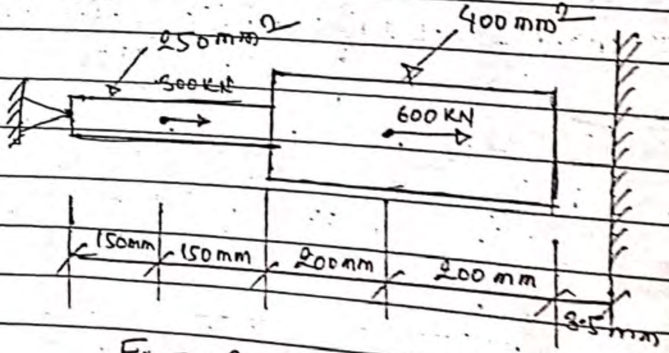
$$10^5 \left(3.333*0 - 3.333*0.654 + 0*0.41 + 0*0 \right) = R_1$$

$$R_1 = -217978.2 = -217.9782 \text{ KN}$$ Ans

And $$10^5 \left(0*0 + 0*0.654 - 2*0.41 + 2*0 \right) = R_4$$

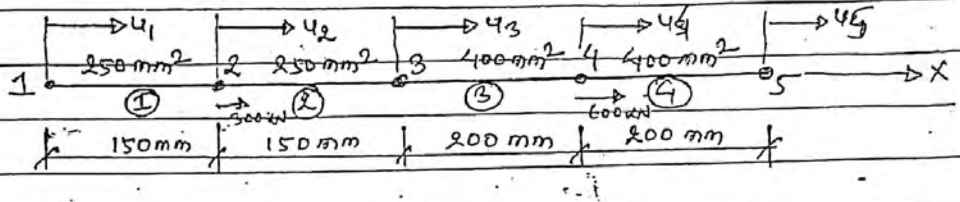
$$R_4 = -82000 \text{ N} = -82 \text{ KN}$$ Ans

QNo.4 Consider the bar as in fig. Determine the nodal displacement, element stresses and Support reactions.



$$E = 200*10^9 \, N/m^2 = 2*10^5 \, N/mm^2$$

Solⁿ 1st modelling of structure and we should 1st determine whether contact occurs between the bar and the wall. To do this, assume that the wall does not exist.



element stiffness matrix

$$[K^1] = \frac{E_1 A_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2*10^5*250}{150}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{array}{cc|c} 1 & 2 & \\ \hline 3.33 & -3.33 & 1 \\ -3.33 & 3.33 & 2 \end{array}$$

$$[K^2] = \frac{E_2 A_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2*10^5*250}{150}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{array}{cc|c} 2 & 3 & \\ \hline 3.33 & -3.33 & 2 \\ -3.33 & 3.33 & 3 \end{array}$$

$$[K^3] = \frac{E_3 A_3}{l_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2*10^5*400}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{array}{cc|c} 3 & 4 & \\ \hline 4 & -4 & 3 \\ -4 & 4 & 4 \end{array}$$

$$[K^4] = \frac{E_4 A_4}{l_4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2*10^5*400}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{array}{cc|c} 4 & 5 & \\ \hline 4 & -4 & 4 \\ -4 & 4 & 5 \end{array}$$

On Assembling, Global stiffness matrix

$$[K] = 10^5 \begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & \\ \hline 3.33 & -3.33 & 0 & 0 & 0 & 1 \\ -3.33 & 6.66 & -3.33 & 0 & 0 & 2 \\ 0 & -3.33 & 7.33 & -4 & 0 & 3 \\ 0 & 0 & -4 & 8 & -4 & 4 \\ 0 & 0 & 0 & -4 & 4 & 5 \end{array}$$

$$\{F\} = \begin{Bmatrix} ① \\ 3 \\ 0 \\ 6 \\ ⑤ \end{Bmatrix} \times 10^5 \text{ N}$$

Global equilibrium

$$[K]\{u\} = F$$

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 10^5 & \cancel{3.33} & \cancel{-3.33} & \cancel{0} & \cancel{0} & \cancel{0} \\ & -3.33 & 6.66 & -3.33 & 0 & 0 \\ & 0 & -3.33 & 7.33 & -4 & 0 \\ & 0 & 0 & -4 & 8 & -4 \\ & 0 & 0 & 0 & -4 & 4 \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} ① \\ 3 \\ 0 \\ 6 \\ 0 \end{Bmatrix} \times 10^5 \quad \begin{array}{c} \cancel{1} \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

Applying boundary condition $u_1 = 0$,

Then, we get following eqn.

$$6.66 u_2 - 3.33 u_3 = 3 \quad — ①$$

$$-3.33 u_2 + 7.33 u_3 - 4 u_4 = 0 \quad — ②$$

$$-4 u_3 + 8 u_4 - 4 u_5 = 6 \quad — ③$$

$$-4 u_4 + 4 u_5 = 0 \quad — ④$$

From ④ $u_4 = u_5$

From ③ $-4u_3 + 8u_4 - 4u_4 = 6$

$$-4u_3 + 4u_4 = 6 \quad — ⑤$$

From ① $u_2 = \frac{3+3.33u_3}{6.66}$ — ⑥

Substituting it into ② ..

$$-3.33\left(\frac{3+3.33u_3}{6.66}\right) + 7.33u_3 - 4u_4 = 0$$

$$-1.5 - 1.667u_3 + 7.33u_3 - 4u_4 = 0$$

$$+5.667u_3 - 4u_4 = 1.5 \quad — ⑦$$

Solving ⑤ & ⑦

$u_3 = 4.5$ mm

$u_4 = 6.009$ mm $= u_5$

∴ displacement at node 5, $u_5 = 6.009 > 3.5$ mm. From this result, we see that contact occur. The problem has to be re-solved since the boundary conditions are different.

The displacement at node 5 is specified to be 3.5 mm.

So boundary condition $u_1 = 0$

$\boxed{u_5 = 3.5 \text{ mm.}}$

Now, Again solving the above eqn.

$$\cancel{10^5}\begin{bmatrix} \cancel{3.33} & \cancel{-3.33} & \cancel{0} & \cancel{0} & \cancel{0} \\ -3.33 & 6.66 & -3.33 & 0 & 0 \\ 0 & -3.33 & 7.33 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & -4 & 4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ 3.5 \end{Bmatrix} = \begin{Bmatrix} Ⓡ \\ 3 \\ 0 \\ 6 \\ 0 \end{Bmatrix} \cancel{10^5}$$

Applying $u_1 = 0$, We get following eqn

$6.66u_2 - 3.33u_3 = 3$ — ①

$-3.33u_2 + 7.33u_3 - 4u_4 = 0$ — ②

$-4u_3 + 8u_4 - 4*3.5 = 6 \Rightarrow -4u_3 + 8u_4 = 20$ — ③

on solving

$u_2 = 2.0193\ mm$, $u_3 = 3.1378\ mm$

$u_4 = 4.0689\ mm$ & $u_5 = 3.5\ mm$

$$\{u\} = \{0 \quad 2.0193 \quad 3.1378 \quad 4.0689 \quad 3.5\}^T$$

Stresses

$$\sigma^1 = E\{B\}\{u\}^1 = 2*10^5 * \frac{1}{150}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}0\\ 2.0193\end{Bmatrix} = 2692.4\ N/mm^2$$

$$\sigma^2 = 2*10^5 * \frac{1}{150}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}2.0193\\ 3.1378\end{Bmatrix} = 1491.33\ N/mm^2$$

$$\sigma^3 = 2*10^5 * \frac{1}{200}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}3.1378\\ 4.0689\end{Bmatrix} = 931.10\ N/mm^2$$

$$\sigma^4 = 2*10^5 * \frac{1}{200}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}4.0689\\ 3.5\end{Bmatrix} = -568.9\ N/mm^2$$

Suppost Reaction

p.t.o

Again writting the element eq^n

$$\cancel{10^5}\begin{bmatrix}3.33 & -3.33 & 0 & 0 & 0\\ -3.33 & 6.66 & -3.33 & 0 & 0\\ 0 & -3.33 & 7.33 & -4 & 0\\ 0 & 0 & -4 & 8 & -4\\ 0 & 0 & 0 & -4 & 4\end{bmatrix}\begin{Bmatrix}0\\ 2.0193\\ 3.1378\\ 4.0689\\ 3.5\end{Bmatrix} = \begin{Bmatrix}R_1\\ 3\\ 0\\ 6\\ \cancel{6}R_2\end{Bmatrix}\cancel{*10^5}$$

$$-3.33*2.0193 = R_1 \qquad \therefore R_1 = -6.7243*10^5\,N$$
$$= 6.7243*10^5\,N\,(\leftarrow)$$

$$-4*4.0689+4*3.5 = R_2$$
$$R_2 = -2.2756*10^5\,N$$
$$= 2.2756*10^5\,N\,(\leftarrow)$$

⑤ A 4m long 300 mm * 300 mm concrete member is vertically hanging. Investigate the state of deformation and stress in the member subjected to gravity using linear bar element

displacement $u(x) = \frac{\rho g}{E}\left(lx-\frac{x^2}{2}\right)$

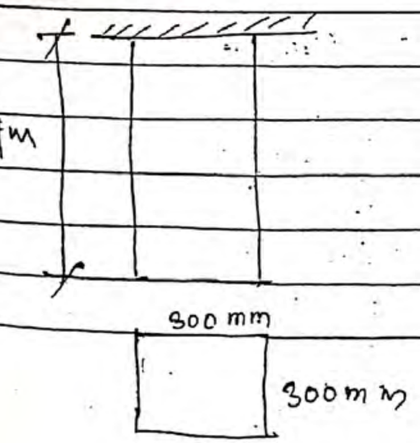
Stress $\sigma_x = \rho g(l-x)$

Assume:

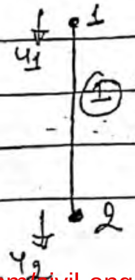
$E = 20\,GPa = 2*10^4\,N/mm^2$

$\rho g = 25\,kN/m^3$

$= 25*10^{-6}\,N/mm^3$



Sol^n One element modelling



$A = (300*300)\,mm^2$

$l = 4000\,mm$

element stiffness matrix

$$[K] = \frac{A_1 E}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{9*10^4 * 2*10^4}{4000}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 4.5*10^5\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} N/$$

Body force vector

$$f = \frac{fAl}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix} = \frac{(25*10^{-6})*(9*10^4)*(4000)}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}$$

$$= 4500\begin{Bmatrix}1\\ 1\end{Bmatrix} N$$

We have

$$\{F\} = [K]\{u\}$$

$$4500\begin{Bmatrix}1\\ 1\end{Bmatrix} = 4.5*10^5\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}$$

Applying boundary condition $u_1 = 0$,

then $4.5*10^5 * u_2 = 4500$

$u_2 = 0.01$ mm.

And By using eqn $u(x) = \frac{\rho g}{E}\left(lx - x^2/2\right)$

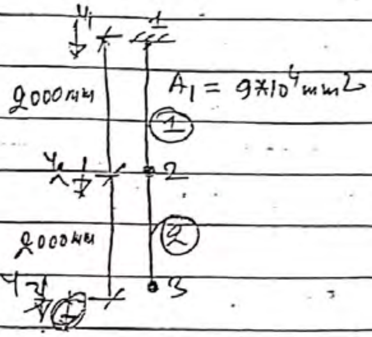
Now, displacement at x=4000mm $= \frac{25*10^{-6}}{2*10^4}\left(4000^{2}\cancel{*} - \frac{4000^2}{2}\right)$

$= 0.01$ mm ~~ff~~ Statisfied

Stress $(\sigma) = E[B]\{u\} = 2*10^4 * \frac{1}{4000}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}0\\ 0.01\end{Bmatrix}$

$= 0.05 N/mm^2$

b) <u>using</u> <u>two</u> element



element stiffness matrix

$$[K^1] = \frac{A_1E_1}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{9*10^4*2*10^4}{2000}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 9*10^5 \begin{array}{cc|c} 1 & 2 & \\ \hline 1 & -1 & 1\\ -1 & 1 & 2\end{array}$$

$$[K^2] = \frac{A_2E_2}{l_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{9*10^4*2*10^4}{2000}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 9*10^5 \begin{array}{cc|c} 2 & 3 & \\ \hline 1 & -1 & 2\\ -1 & 1 & 3\end{array}$$

On Assembling global stiffness matrix,

$$[K] = 9*10^5 \begin{array}{ccc|c} 1 & 2 & 3 & \\ \hline 1 & -1 & 0 & 1\\ -1 & 2 & -1 & 2\\ 0 & -1 & 1 & 3\end{array}$$

Body force vector,

$$\{f^1\} = \frac{f_1A_1l_1}{2}\begin{Bmatrix}1\\1\end{Bmatrix} = \frac{(25*10^{-6})(9*10^4)*2000}{2}\begin{Bmatrix}1\\1\end{Bmatrix} = 2250\begin{Bmatrix}1\\1\end{Bmatrix}\begin{matrix}1\\2\end{matrix}$$

$$\text{||ly } \{f^2\} = 2250\begin{Bmatrix}1\\1\end{Bmatrix}\begin{matrix}2\\3\end{matrix}$$

$$\text{load vector } \{F\} = 2250\begin{Bmatrix}1\\2\\1\end{Bmatrix}\begin{matrix}1\\2\\3\end{matrix}$$

Now, $[K]\{u\} = F$

$$9*10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 2250 \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix}$$

We have, boundary Condition $u_1 = 0$

then

$$9*10^5 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = 2250 \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$$

On solving

$u_2 = 7.5*10^{-3}$ mm

$u_3 = 0.01$ mm.

$\{u\} = \begin{Bmatrix} 0 \\ 7.5*10^{-3} \\ 0.01 \end{Bmatrix}$ using formula

$$u(x) = \frac{\rho g}{E}\left(lx - \frac{x^2}{2}\right) = \frac{25*10^{-6}}{2*10^4}\left(4000x - \frac{x^2}{2}\right)$$

$$= 1.25*10^{-9}(4000x - 0.5x)$$

at node 1, $x = 0$, $u_1 = 0$ mm

at node 2, $x = 2000$mm, $u_2 = 7.5*10^{-3}$ mm

at node 3, $x = 4000$mm, $u_3 = 0.01$ mm

Stresses

$\sigma^1 = E[B]^1\{u\} = (2*10^4)*\frac{1}{2000}[-1\ 1]\begin{Bmatrix} 0 \\ 7.5*10^{-3} \end{Bmatrix} = 0.075$ N/mm²

$\sigma^2 = E[B]^2\{u\} = (2*10^4)*\frac{1}{2000}[-1\ 1]\begin{Bmatrix} 7.5*10^{-3} \\ 10*10^{-3} \end{Bmatrix} = 0.025$ N/mm²

stress $\sigma(x) = \rho g(L-x) = 25*10^{-6}(4000-x)$

In C.G. of element ①	$x = 1000$ mm
In C.G. of element ②	$x = 3000$ mm

$\sigma^{①} = 25*10^{-6}(4000-1000) = 0.075$

$\sigma^2 = 25*10^{-6}(4000-1000) = 0.025$

OK//

Temperature effect

Element load vector due to temperature change

$$\boxed{\{f_T^e\} = AE\alpha\Delta t\begin{Bmatrix}-1\\1\end{Bmatrix}}$$

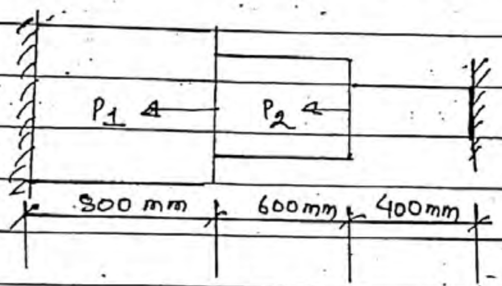
strain due to temp. change $\boxed{(\varepsilon_0) = \alpha\Delta t}$

Stress in each element

$$\text{Total stress } \sigma = E[B]\{q\} - E\alpha\Delta t$$

$$\text{" strain } (e) = \cancel{E}[B]\{q\} - \alpha\Delta t$$

QNO.6 The structure shown in fig is subjected to an increase in temperature, $\Delta t = 80°C$, Determine displacements, stresses and support reaction.



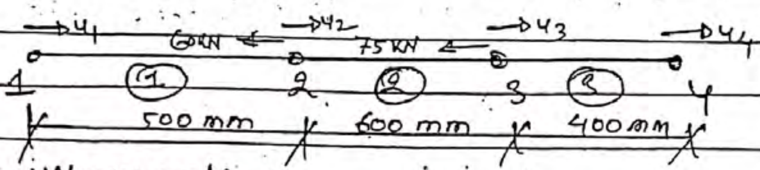
$P_1 = 60$ KN

$P_2 = 75$ KN

$\Delta t = 80°C$

Bronze	Aluminium	Steel
$A = 2400\,mm^2$	$1200\,mm^2$	$600\,mm^2$
$E = 83\,Gpa$	$70\,Gpa$	$200\,Gpa$
$\alpha = 18.9*10^{-6}/°C$	$23*10^{-6}/°C$	$11.7*10^{-6}/°C$

Sol^n Modelling of structure.



element stiffness matrix

$$[K^1] = \frac{A_1E_1}{l_1}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = \frac{2400*83*10^3}{500}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = 10^5\begin{array}{ccc} 1 & 2 & \\ \left[\begin{matrix}2.49 \\ -2.49\end{matrix}\right. & \left.\begin{matrix}-2.49 \\ 2.49\end{matrix}\right] & \begin{matrix}1\\2\end{matrix}\end{array}$$

$$[K^2] = \frac{A_2E_2}{l_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{1200*70*10^3}{600}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 10^5\left[\begin{array}{cc|c}1.4 & -1.4 & 2\\ -1.4 & 1.4 & 3\end{array}\right]$$

$$[K^3] = \frac{A_3E_3}{l_3}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{600*200*10^3}{400}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 10^5\left[\begin{array}{cc|c}3 & 4 & \\ \hline 3 & -3 & 3\\ -3 & 3 & 4\end{array}\right]$$

On Assembling, Global stiffness matrix:

$$[K] = 10^5\left[\begin{array}{cccc|c}1 & 2 & 3 & 4 & \\ \hline 2.49 & -2.49 & 0 & 0 & 1\\ -2.49 & 3.89 & -1.4 & 0 & 2\\ 0 & -1.4 & 4.4 & -3 & 3\\ 0 & 0 & -3 & 3 & 4\end{array}\right]$$

Load vector due to externally applied load.

$$\{P\} = \left\{\begin{array}{c|c}0 & 1\\ -60000 & 2\\ -75000 & 3\\ 0 & 4\end{array}\right.$$

Load vector due to temperature change

For element ①

$$\{f_T^1\} = A_1E_1\alpha_1\Delta t_1\begin{Bmatrix}-1\\ 1\end{Bmatrix} = 2400*83*10^3*18.9*10^{-6}*80\begin{Bmatrix}-1\\ 1\end{Bmatrix}$$

$$= \begin{Bmatrix}-301.19\\ 301.19\end{Bmatrix}10^3 \begin{array}{c}1\\ 2\end{array}$$

$$\begin{Bmatrix} \\ 1 \end{Bmatrix} = 10 \begin{Bmatrix} -112.32 \\ 112.32 \end{Bmatrix} \begin{matrix} 3 \\ 4 \end{matrix}$$

On Assembling load vector due to temperature change for whole str.

$$\{f_T\} = 10^3 \begin{Bmatrix} -301.19 \\ 146.63 \\ 42.24 \\ 112.32 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

Now, Global load vector

$$\{F\} = 10^3 \begin{Bmatrix} 0-301.19+R_1 \\ -60+146.63 \\ -75+42.24 \\ 0+112.32+R_4 \end{Bmatrix} = 10^3 \begin{Bmatrix} R_1-301.19 \\ 86.63 \\ -32.76 \\ 112.32+R_4 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

Ato $[K]\{u\} = \{F\}$

$$0^5 \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 2.49 & -2.49 & 0 & 0 \\ -2.49 & 3.89 & -1.4 & 0 \\ 0 & -1.4 & 4.4 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = 10^3 \begin{Bmatrix} R_1-301.19 \\ 86.63 \\ -32.76 \\ 112.32+R_4 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

Applying boundary condition, $u_1 = u_4 = 0$

$$10^5\begin{bmatrix}3.89 & -1.40\\ -1.40 & 4.4\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}86.63\\ -32.76\end{Bmatrix}10^3$$

On solveing

$$\begin{Bmatrix}u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}0.2212\\ -4.06*10^{-3}\end{Bmatrix}mm$$

Stresses.

In bronze, element 1.

$$\sigma^1 = E\{B\}\{u\}-\underline{E\alpha\Delta t} = E\left[\frac{1}{800}[-1\ \ 1]\begin{Bmatrix}0\\ 0.2212\end{Bmatrix}-(18.9*10^{-6}*80)\right]$$

$$= 83*10^3*(-1.02355*10^{-3}) = -102.55\ N/mm^2$$

$$\sigma^2 = 70*10^3\left[\frac{1}{600}[-1\ \ 1]\begin{Bmatrix}0.2212\\ -4.06*10^{-3}\end{Bmatrix}-(23*10^{-6}*80)\right]$$

$$= -155.08\ N/mm^2 \ \underline{Ans}$$

$$\sigma^3 = 200*10^3\left(\frac{1}{400}[-1\ \ 1]\begin{Bmatrix}-4.06*10^{-3}\\ 0\end{Bmatrix}-(11.7*10^{-6}*80)\right)$$

$$= -185.17\ N/mm^2 \ \underline{Ans}$$

Suppost reaction.
$$10^5\begin{bmatrix}2.49 & -2.49 & 0 & 0\\ -2.49 & 3.89 & -1.4 & 0\\ 0 & -1.4 & 44 & -3\\ 0 & 0 & -3 & 3\end{bmatrix}\begin{Bmatrix}0\\ 0.2212\\ -4.06*10^{-3}\\ 0\end{Bmatrix}=10^3\begin{Bmatrix}R_1-30\\ -86.6\\ -32\\ 112\end{Bmatrix}$$

On solveing $R_1 = 246.1112\ KN$

$R_2 = -11$[illegible]$2\ KN$

BAR ELEMENT

The term bar is used in solid and structural mechanics to mean a structural element that carries only axial loads (tensile as well as compressive)

Formulation of finite element characterstics of bar element

Consider a bar of length (l) in which a uniaxial coordinate system x with its origin arbitrarily placed at a some distance from the left end is fixed as shown in following figure. Denoting axial displacement at any position along the length of bar as $u(x)$, We define nodes 1 and 2 at each ~~[illegible]~~ end as shown and introduce the nodal displacements u_1 at $x = x_1$ and u_2 at $x = x_2$.

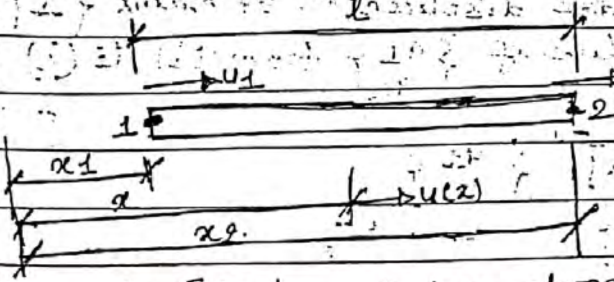


Fig :- A bar element with global coordinate system and nodal displacement notation

Let displacement field in bar is approximated by linear polynomial.

No. of generalized coordinate = no. of degree of freedom

$u(x) = a1 + a2x$

$$u(x) = \begin{bmatrix} 1 & x \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \text{ ——— } \langle i \rangle$$

At nodes

$$u_1 = a_1 + a_2 x_1$$

$$u_2 = a_1 + a_2 x_2$$

or, $$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

Denoting $[A] = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [A] \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

or, $$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \text{ ——— } \langle ii \rangle$$

or, $$\{a\} = [A]^{-1} \{\underline{u}\}$$

denoting nodal displacement by matrix $\{\underline{u}\}$

Substituting the value of $\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$ from (ii) to (i)

$$u(x) = \begin{bmatrix} 1 & x \end{bmatrix} [A]^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

or, $$u(x) = [N] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \text{ ——— } (iii)$$

Where $[N] = \begin{bmatrix} 1 & x \end{bmatrix} [A]^{-1}$ which is called the linear finite element approximation ~~fun~~ function or interpolation function or shape function or bending function.

$$[N] = \begin{bmatrix} 1 & x \end{bmatrix} [A]^{-1}$$

$$= \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}^{-1}$$

$$\text{or, } [N] = \begin{bmatrix} 1 & x \end{bmatrix} \cdot \frac{1}{x_2 - x_1} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix}$$

$$\text{or, } [N] = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} \frac{x_2}{x_2 - x_1} & \frac{-x_1}{x_2 - x_1} \\ \frac{-1}{x_2 - x_1} & \frac{1}{x_2 - x_1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x_2}{x_2 - x_1} - \frac{x}{x_2 - x_1} & \frac{-x_1}{x_2 - x_1} + \frac{x}{x_2 - x_1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(x_2 - x)}{(x_2 - x_1)} & \frac{(x - x_1)}{(x_2 - x_1)} \end{bmatrix} = \begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix}$$

Here, $N_1(x) = \frac{x_2 - x}{x_2 - x_1}$ $N_2(x) = \frac{x - x_1}{x_2 - x_1}$ —— ⓘⓥ

From (iii)

$$u = \begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$u = N_1(x) u_1 + N_2(x) u_2$$

Shape function [N] have some intresting properties.

At nodes;

From (iv), N_1 at $x = x_1$, $= \frac{x_2 - x_1}{x_2 - x_1} = \cancel{0}1$

N_2 at $x = x_1 = \frac{x_1 - x_1}{x_2 - x_1} = 0$

Similarly,

N_1 at $x = x_2 = \frac{x_2 - x_2}{x_2 - x_1} = 0$

$$N_2 \text{ at } x = x_2 = \frac{x_2 - x_1}{x_2 - x_1} = 1$$

In general, N_i is unity at the i^{th} node and zero at the other node.

→ Another property of shape function is that their sum is unity.

$$N_1 + N_2 = \frac{x_2 - x}{x_2 - x_1} + \frac{x - x_1}{x_2 - x_1} = 1$$

$$\text{i.e. } \sum_{i=1}^{n} N_i(x) = 1$$

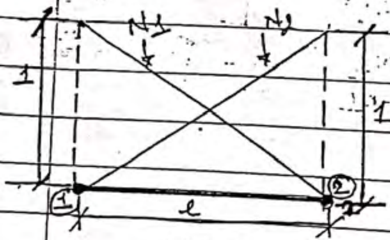


fig: shape function

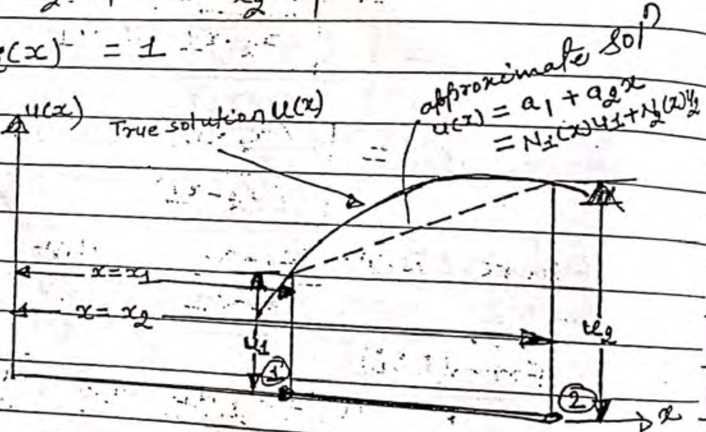


fig:- Linear finite element approximation

The shape function N_i in eq^n (iv) were derived in terms of global coordinate system. Now, let origin is fixed at node ① of the element as shown in fig.

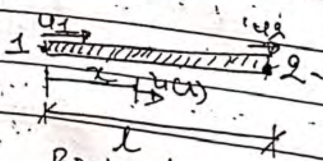


Fig:- bar element in local coordinate system, here, $x_1 = 0$, $x_2 = l$

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1} = \frac{l - x}{l} = 1 - \frac{x}{l}$$

$$N_2(x) = \frac{x - x_1}{x_2 - x_1} = \frac{x - 0}{l} = \frac{x}{l}$$

This coordinate system is termed as local or element coordinate system.

The degree (or order) of polynomial equation can be increased to improve accuracy.

Try, this with following quadratic approximation of u(x)

$$u(x) = a_1 + a_2 x + a_3 x^2$$

Strain in bar

We have strain - displacement relation.

$$\epsilon = \frac{du}{dx}$$

$$= \frac{d}{dx}\begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{d}{dx}\begin{bmatrix} \frac{x_2 - x}{x_2 - x_1} & \frac{x - x_1}{x_2 - x_1} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{x_2 - x_1} & \frac{1}{x_2 - x_1} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{x_2 - x_1}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\epsilon = [B]\{\underline{u}\}$$

Where $[B] = \frac{1}{x_2 - x_1}\begin{bmatrix} -1 & 1 \end{bmatrix} = \frac{1}{l}\begin{bmatrix} -1 & 1 \end{bmatrix}$

$\{\underline{u}\}$ = column matrix of nodal displacements.

https://civinnovate.com/civil-engineering-notes/

$\sigma = E\varepsilon$

$\boxed{\sigma = E[B]\{u\}}$ ... Where $\varepsilon = [B]\{u\}$

Direct stiffness matrix for bar element

From strength of material, the deflection δ of a bar of length L and uniform cross-sectional area A when subjected to axial load P is given by

$$\delta = \frac{PL}{AE} \quad \text{—— (a)}$$

Where E = Young's modulus of elasticity of material

From using eq^n (a), we obtain the equivalent spring constant of bar is

$$K = \frac{P}{\delta} = \frac{AE}{L} \quad \text{—— (b)}$$

In uniaxial loading as in the bar element, we need consider only normal strain component defined as

$$\text{strain, } \varepsilon = \frac{\text{Change in length (elongation)}}{\text{original length}}$$

$$\varepsilon = \frac{du}{dx} = \frac{u_2 - u_1}{L}$$

axial stress $(\sigma) = E\varepsilon = \frac{E(u_2 - u_1)}{L}$

and axial force

$$P = \sigma * A = \frac{E(u_2 - u_1)}{L} * A = \frac{AE(u_2 - u_1)}{L} \quad \text{—— (c)}$$

~~When~~ if equation (c) has positive sign, the element is in tension and nodal force f_2 must be in positive coordinate direction while nodal force f_1 must be equal and opposite to the f_2 for equilibrium

$$f_1 = -\frac{AE}{L}(u_2-u_1) = \frac{AE}{L}(u_1-u_2) \quad — ⓓ$$

$$f_2 = \frac{AE}{L}(u_2-u_1) = \frac{AE}{L}(-u_1+u_2) \quad — ⓔ$$

expressing eqn ⓓ & ⓔ in matrix form.

$$\frac{AE}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{Bmatrix}f_1\\ f_2\end{Bmatrix} \quad — ⓕ$$

We have displacement eqn $[K]\{u\} = \{f\} \rightarrow$ ⓖ

Comparing eqn ⓕ & ⓖ

$$K = \frac{AE}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Which is the ~~eler~~ stiffness matrix for the bar element.

potential energy of an elastic body - eqⁿ

$$U_e = \frac{1}{2}\int_V \{\sigma\}^T\{\varepsilon\}\,dV$$

But $\{\sigma\} = [D]\{\varepsilon\}$ From chapter 3 (Elasticity in solids)

or, $U_e = \frac{1}{2}\int_V \left[[D]\{\varepsilon\}\right]^T \{\varepsilon\}\,dV$ $\quad \because \{[A][B]\}^T = [B]^T[A]^T$

$$= \frac{1}{2}\int_V \{\varepsilon\}^T[D]^T\{\varepsilon\}\,dV$$

but $[D]^T = [D]$ -- Because of symmetry

$$\therefore U_e = \frac{1}{2}\int_V \{\varepsilon\}^T[D]\{\varepsilon\}\,dV$$

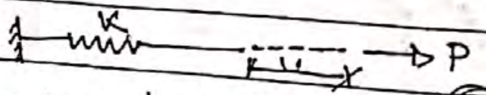
We know $\{\varepsilon\} = [B]\{\underline{u}\}$

$$U_e = \frac{1}{2}\int_V \left[[B]\{\underline{u}\}\right]^T [D]\{\varepsilon\}\,dV$$

$$= \frac{1}{2}\int_V \{\underline{u}\}^T[B]^T[D][B]\{\underline{u}\}\,dV$$

$$= \frac{1}{2}\{\underline{u}\}^T\left[\int_V [B]^T[D][B]\,dV\right]\{\underline{u}\} \quad — ⓘ$$

We know for spring system.



potential energy $u = \frac{1}{2}(u\cdot K)\cdot u$ — (ii)

force

Comparing eqn ⓘ + (ii), eqn ⓘ becomes

$$U_e = \frac{1}{2}\{\underline{u}\}^T[K]\{\underline{u}\}$$

$$[K] = \int_V [B]^T [D] [B] \, dV$$

In this case $[D] = E \quad \because \sigma = [D]\,\varepsilon = E\varepsilon$

E = Young's modulus of elasticity

& B is also constant

$$[K] = [B]^T E [B] \int_V dV$$

We know, $[B] = \frac{1}{x_2 - x_1}\begin{bmatrix} -1 & 1 \end{bmatrix} = \frac{1}{l}\begin{bmatrix} -1 & 1 \end{bmatrix}$

& $\int_V dV = V = Al$

$$[K] = AL\,[B]^T E\,[B]$$

$$= AEl \; \frac{1}{l}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \frac{1}{l}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$\boxed{[K] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

Force terms

Consider ~~the~~ element body force term from total potential energy

$$\int_V \{u\}^T \{f\} \, dv \qquad \{u\} = [N]\{\underline{u}\}$$

unit

$$\{u\}^T = \left[[N]\{\underline{u}\}\right]^T = \{\underline{u}\}^T [N]^T$$

$$\Rightarrow \int_V \{\underline{u}\}^T [N]^T \{f\} \, dv$$

$$= \{\underline{u}\}^T \int_V [N]^T \{f\} \, dv$$

$= \{\underline{u}\}^T . \{f^e\}$ Where $\{f^e\}$ = element body force vector

$$\{f^e\} = \int_V [N]^T \{f\}\, dv$$

For this case, $\{f\} = f$, $[N]^T$ ~~$= \frac{1}{x_2 - x_1}$~~

$$[N] = \frac{1}{x_2 - x_1}\begin{bmatrix} x_2 - x & x - x_1 \end{bmatrix}$$

$$[N]^T = \frac{1}{x_2 - x_1}\begin{bmatrix} x_2 - x \\ x - x_1 \end{bmatrix}$$

$$\{f^e\} = f \int_V [N]^T\, dv$$

$$= f \int_0^l \frac{1}{x_2 - x_1}\begin{bmatrix} x_2 - x \\ x - x_1 \end{bmatrix} A\, dx$$

$$= \frac{fA}{l}\int_0^l \begin{bmatrix} x_2 - x \\ x - x_1 \end{bmatrix} dx \quad \text{Where } x_2 - x_1 = l$$

$$= \frac{fA}{l}\begin{bmatrix} x_2 l - \frac{l^2}{2} \\ l^2/2 \quad - x_1 l \end{bmatrix} = \frac{fA \times l}{l}\begin{bmatrix} x_2 - l/2 \\ l/2 - x_1 \end{bmatrix}$$

if the origin at left end, i.e. $x_1 = 0,\ x_2 = l$

$$\{f^e\} = fA \begin{bmatrix} l - l/2 \\ l/2 - 0 \end{bmatrix} = fA \begin{bmatrix} l/2 \\ l/2 \end{bmatrix}$$

$$\boxed{\{f^e\} = \frac{fAl}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$f \to$ unit weight of material

⇒ SHAPE FUNCTION IN TERMS OF NATURAL OR INTRINSIC COORDINATE SYSTEM (ξ)

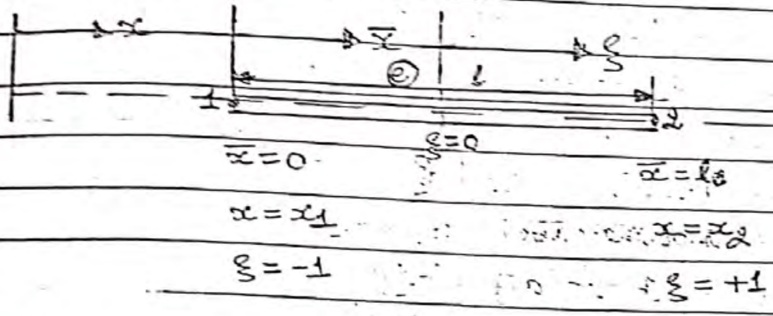


Fig:- Global coordinate x, local coordinate $\bar{x}$ and natural coordinate ξ

From fig

When $x=x_1$, $\xi=-1$

When $x=x_2$, $\xi=+1$

When $x=x$, $\xi=\xi$

From linear Enterpolation,

$$\xi-(-1)=\frac{+1-(-1)}{x_2-x_1}(x-x_1)$$

$$\boxed{\xi=\frac{2}{x_2-x_1}(x-x_1)-1}$$

The coordinate ξ is called normal coordinate or natural coordinate and its values always lie between -1 and $+1$, with its origin at the centre of element.

SHAPE FUNCTION

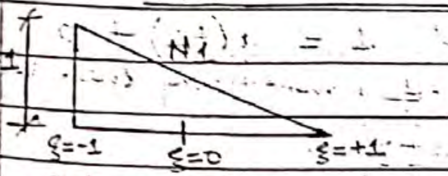


Fig:- SHAPE FUNCTION (N_1)

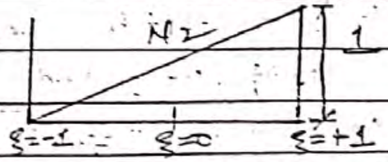


Fig:- SHAPE FUNCTION N_2

$$\left.\begin{array}{ll}\text{When } \xi = -1, & N_1 = 1 \\ \text{When } \xi = +1 & N_1 = 0\end{array}\right\} ①$$

Hence, N_1 is function ξ

Shape function is linear,

Hence,

$$N_1(\xi) = a\xi + b \quad — ②$$

Imposing eqn ① condition

$$1 = a(-1) + b$$

and $0 = a(+1) + b$

on solving

$b = \frac{1}{2}$ and $a = -\frac{1}{2}$

Hence, ~~at~~ substituting these values in ②

$$N_1(\xi) = -\frac{1}{2}\xi + \frac{1}{2} = \frac{1-\xi}{2}$$

SHAPE FUNCTION N_2

$$N_2(\xi) = a\xi + b \quad — ⑤$$

When $\xi = -1$, $N_2 = 0$ and $\xi = +1$, $N_2 = 1$

Imposing these condition

$0 = a(-1) + b$ and $1 = a(+1) + b$

on solving $b = 1/2$ and $a = \frac{1}{2}$, substituting these value in ⑤

$$\boxed{N_2(\xi) = \frac{1}{2}\xi + \frac{1}{2} = \frac{1+\xi}{2}}$$

or, We have already derived that shape function in terms of global coordinate system, i.e.

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1} \quad \text{and} \quad N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

put, $x_2 = +1$,

$x_1 = -1$,

and $x_2 - x_1 = +1-(-1) = 2$, then

$$N_1(\xi) = \frac{1-\xi}{2} \qquad N_2(\xi) = \frac{\xi-(-1)}{2} = \frac{1+\xi}{2}.$$

Also we have,

$$N_1 = \frac{x_2 - x}{x_2 - x_1}$$

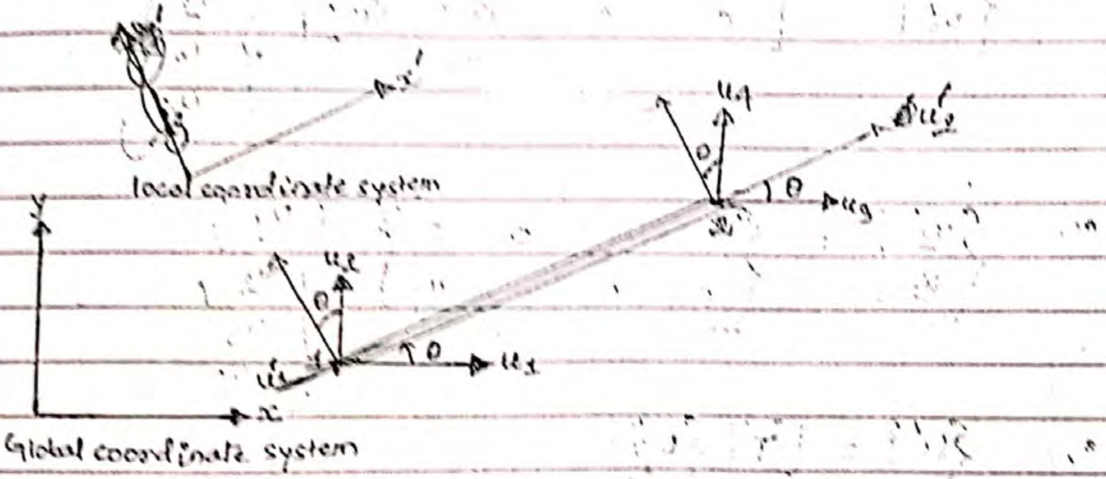
or, $N_1 x_2 - N_1 x_1 = x_2 - x$

or, $x = x_2(1-N_1) + N_1 x_1$

$x = x_2 N_2 + N_1 x_1 \qquad (\because N_1 + N_2 = 1)$

$\boxed{x = N_1 x_1 + N_2 x_2}$ → Another ~~to~~ transformation form x to ξ

Consider a uniform bar element with constant EA and oriented at an angle θ, measured counterclockwise, from the +ve x-axis. in local and global coordinate system as shown in fig.



Fig(a) truss element

The local coordinate system consists of x' axis which runs along the element from node 1 toward node 2. All quantities in the local coordinate system will be denoted by a prime ('). The global x-y coordinate system is fixed and does not depend upon the orientation of elements. In global coordinate system, every node has two degrees of freedom (dofs).

Let u_1 and u_2 be the displacement of node 1 in global coordinate system and u_3 & u_4 be the displacement of node 2 in global coordinate direction.

u_1' & u_2' be the displacement of node 1 and node 2 in local coordinate direction respectively.

Now, → displacement in local coordinate system

$$\{u'\} = \begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix}$$

From fig, $u_1' = u_1\cos\theta + u_2\sin\theta$

$u_2' = u_3\cos\theta + u_4\sin\theta$

or
$$\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

or,
$$\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \boxed{\begin{Bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{Bmatrix}} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

or,
$$\boxed{\{u'\} = [T]\{u\}}$$

Where, $\{u'\}$ = local displacements matrix in local coordinate direction (ie-along the axes of member)

$[T]$ = Transformation matrix

$\{u\}$ = displacement matrix in global coordinate direction

Where $l = \cos\theta$,

$m = \sin\theta$

Member forces

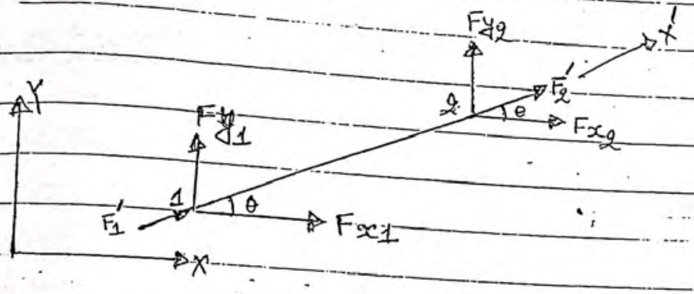


fig (b) Member forces in local and global coordinate system.

Resolving F_1' and F_2' in global coordinate direction

$F_{x1} = F_1' \cos\theta$ $\qquad$ $F_{y1} = F_1' \sin\theta$

$F_{x2} = F_2' \cos\theta$ $\qquad$ $F_{y2} = F_2' \sin\theta$

Writting these equation in matrix form

$$\begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} \begin{Bmatrix} F_1' \\ F_2' \end{Bmatrix}$$

or $\boxed{\{F\} = [T]^T \{F'\}}$ Where, $[T]^T = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix}$

$\{F\}$ = global member force matrix

$[T]^T$ = Transpose of transformation matrix

$\{F'\}$ = local member force matrix

ELEMENT STIFFNESS MATRIX

The truss element is a 'one dimensional element (bar element) when viewed in the local coordinate system. This observation allows us to use previously developed results for bar element.

The element stiffness matrix for a truss element in local coordinate system is given by

$$K' = \frac{AE}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

In local coordinate system

$[K']\{u'\} = \{F'\}$ — ⟨i⟩

In global coordinate system

$[K]\{u\} = \{F\}$ —⟨ii⟩

From previous, we know,

$\{u'\} = [T]\{u\}$ putting this in eqⁿ ①

$[K'][T]\{u\} = \{F'\}$

Premuiteplying both sides by $[T]^T$, we get

$$[T]^T[K'][T]\{u\} = [T]^T\{F'\}$$

$$[T]^T[K'][T]\{u\} = \{F\} \text{ —(iii) } \left(\because \{F\} = [T]^T\{F'\}\right)$$

Comparing ⟨iii⟩ and ⟨ii⟩

$$[K] = [T]^T[K'][T] \text{ — (iv)}$$

$[K]$ = global stiffness matrix

substituting value of $[T]^T$ and $[T]$ in (iv) from above,

$$[K] = \begin{bmatrix} l & 0\\ m & 0\\ 0 & l\\ 0 & m\end{bmatrix}\frac{AE}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix} l & m & 0 & 0\\ 0 & 0 & l & m\end{bmatrix}$$

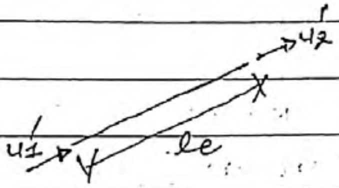
$$[K] = \frac{AE}{l} \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \begin{bmatrix} l & m & -l & -m \\ -l & -m & l & m \end{bmatrix}$$

$$\boxed{[K] = \frac{AE}{L} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}}$$

[illegible]

$$[K] = \frac{AE}{L} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

Stress calculation



$$\sigma = E\varepsilon$$

$$= E\,\frac{u_2' - u_1'}{le}$$

$$\boxed{\sigma = \frac{E}{le} \begin{bmatrix} -1 & +1 \end{bmatrix} \begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix}}$$

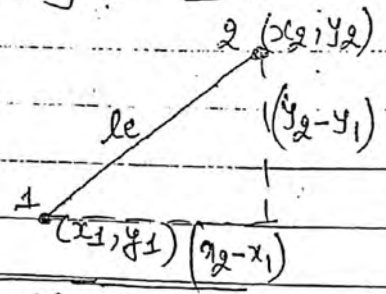
$$= \frac{E}{le} \begin{bmatrix} -1 & 1 \end{bmatrix} \{u'\} = \frac{E}{le} \begin{bmatrix} -1 & 1 \end{bmatrix} [T] \{u\}$$

$$\Rightarrow \sigma = \frac{E}{le} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \{u\}$$

$$\boxed{\sigma = \frac{E}{le} \begin{bmatrix} -l & -m & l & m \end{bmatrix} \{u\}}$$

$$\boxed{= \frac{E}{le} \begin{bmatrix} -c & -s & c & s \end{bmatrix} \{u^*\}}$$

Formulas calculating l and m



$$le = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$$l = \cos\theta = \frac{x_2-x_1}{le}$$

$$m = \sin\theta = \frac{y_2-y_1}{le}$$

In general

For i and i+1 node,

$$l = \cos\theta = \frac{x_{i+1}-x_i}{le}$$

$$m = \sin\theta = \frac{y_{i+1}-y_i}{le}$$

QNo1 The two-element truss in Figure is subjected to external loading as shown. Determine the displacement component of node 3, the reaction forces ~~at no~~ components at nodes 1 and 2 and the element displacements, stresses and forces. The element have modulus of elasticity $E_1 = E_2 = 10*10^6$ lb/in^2 and cross sectional areas $A_1 = A_2 = 1.5$ in^2.

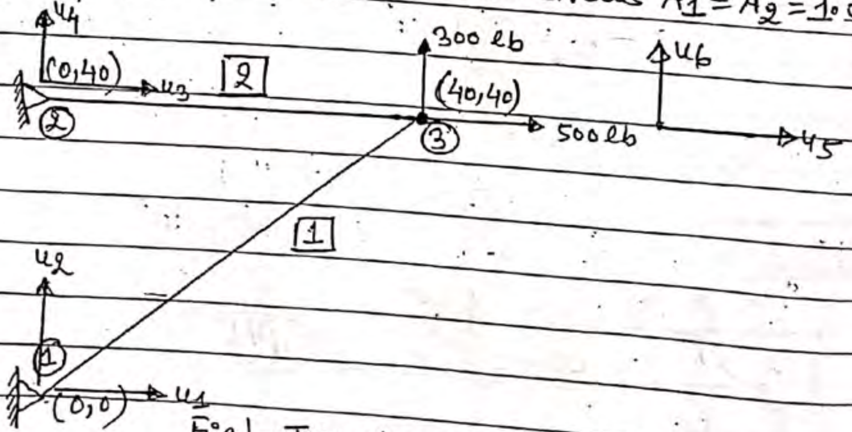


Fig 1:- Two element truss with external loading

steps1: nodal coordinate

Node	x	y
1	0	0
2	0	40
3	40	40

Element Connectivity table and direction Cosines.

Element	Nodes		length	θ	Cosθ	Sinθ

Element connectivity tabe

Element	node 1 Local node [1]	node 2 [2]
1	1	3
2	2	3

direction cosines. (l, m)

Element	le	l	m
1	56·57	$\frac{x_2-x_1}{le} = 0.707$	$\frac{y_2-y_1}{le} = \frac{40-0}{57.57} = 0.707$
2	40	$\frac{x_2-x_1}{le} = \frac{40-0}{\cancel{57.57}\ 40} = 1$	$\frac{y_2-y_1}{le} = \frac{40-40}{40} = 0$

Step2: Generate stiffness matrixes for each element

For element 1

$$[k^1] = \frac{AE}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} = \frac{1.5*10*10^6}{56.57}\begin{bmatrix} 0.707^2 & 0.707*0.707 & -- & -- \\ 0.707*0.707 & 0.707^2 & & \\ & & & -- \\ & & & -- \end{bmatrix}$$

In element 1, displacement q_1 q_2 q_5 q_6

$$[k^1] = 2.65 * 10^5 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}$$

$$[k^1] = 10^5 \begin{array}{c|cccc|c} & 1 & 2 & 5 & 6 & \\ \hline & 1.325 & 1.325 & -1.325 & -1.325 & 1 \\ & 1.325 & 1.325 & -1.325 & -1.325 & 2 \\ & -1.325 & -1.325 & 1.325 & 1.325 & 5 \\ & -1.325 & -1.325 & 1.325 & 1.325 & 6 \end{array}$$

← Global degree freedom

For element 2. (= q_3, q_4, q_5, q_6)

$$[k^2] = \frac{1.05 * 10 * 10^6}{40} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k^2] = 8\ 10^5 \begin{array}{|cccc|l} 3 & 4 & 5 & 6 & \\ \hline 3.75 & 0 & -3.75 & 0 & \cancel{1}\ 3 \\ 0 & 0 & 0 & 0 & \cancel{2}\ 4 \\ -3.75 & 0 & 3.75 & 0 & \cancel{3}\ 5 \\ 0 & 0 & 0 & 0 & \cancel{4}\ 6 \end{array}$$

← Global degree of freedom

Step 3: The structural matrix K is now assembled from the element stiffness matrices.

We have six degree of freedom, the assembled ~~matr~~ stiffness matrix will be of 6*6.

deleted from step 4

$$[K] = 10^5 \begin{array}{c|cccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline & 1.325 & 1.325 & 0 & 0 & -1.325 & -1.325 & 1 \\ & 1.325 & 1.325 & 0 & 0 & -1.325 & -1.325 & 2 \\ & 0 & 0 & 3.75 & 0 & -3.75 & 0 & 3 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ & -1.325 & -1.325 & -3.75 & 0 & 5.075 & 1.325 & 5 \\ & -1.325 & -1.325 & 0 & 0 & 1.325 & 1.325 & 6 \end{array}$$

deleted from step 4

step 4 :- Treatment of boundary condition, Here, $u_1 = u_2 = u_3 = u_4 = 0$ due to support. The rows and columns corresponding to 1, 2, 3, 4 which corresponds the supports are deleted from [K] matrix

the global equilibrium

$$[K]\{u\} = \{F\}$$

external applied loading

reaction

$$10^5 \begin{array}{|cccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline 1.325 & 1.325 & 0 & 0 & -1.325 & -1.325 & 1 \\ 1.325 & 1.325 & 0 & 0 & -1.325 & -1.325 & 2 \\ 0 & 0 & 3.75 & 0 & -3.75 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ -1.325 & -1.325 & -3.75 & 0 & 5.075 & 1.325 & 5 \\ -1.325 & -1.325 & 0 & 0 & 1.325 & 1.325 & 6 \end{array} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} F_1+R_1 \\ F_2+R_2 \\ F_3+R_3 \\ F_4+R_4 \\ 500 \\ 300 \end{Bmatrix}$$

~~Now~~, Reduced finite element equation given as

$$10^5 \begin{bmatrix} 5.075 & 1.325 \\ 1.325 & 1.325 \end{bmatrix} \begin{Bmatrix} u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 500 \\ 300 \end{Bmatrix}$$

on solving,

$u_5 = 5.333 * 10^{-4}$ in. and $u_6 = 1.731 * 10^{-3}$ in

① Calculation of reaction forces

Again writting global equilibrium eqⁿ.

$$10^5 \begin{bmatrix} 1.325 & 1.325 & 0 & 0 & -1.325 & -1.325 \\ 1.325 & 1.325 & 0 & 0 & -1.325 & -1.325 \\ 0 & 0 & 3.75 & 0 & -3.75 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1.325 & -1.325 & -3.75 & 0 & 5.075 & 1.325 \\ -1.325 & -1.325 & 0 & 0 & 1.325 & 1.325 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5.33*10^{-4} \\ 1.731*10^{-3} \end{Bmatrix} = \begin{Bmatrix} F_1+R_1 \\ F_2+R_2 \\ F_3+R_3 \\ F_4+R_4 \\ 500 \\ 300 \end{Bmatrix}$$

Multiplying and writting in eqⁿ form

$$10^5 * \left[-1.325 * 5.33 * 10^{-4} - 1.325 * 1.731 * 10^{-3} \right] = F_1 + R_1$$

Here; $F_1 = 0$ ∵ there is no external loading in coordinate direction 1;

$\therefore R_1 = -300$ lb

① Similarly,

$$10^5 \left[-1.325 * 5.33 * 10^{-4} - 1.325 * 1.731 * 10^{-3} \right] = F_2 + R_2$$

Here, $F_2 = 0$,

$\therefore R_2 = -300$ lb

Similarly

$$10^5 * \left[-3.75 * 5.33 * 10^{-4} \right] = F_3 + R_3$$

Here, $F_3 = 0$,

$\therefore R_3 \approx -200$ lb

Similarly;

$$0 = F_4 + R_4$$

$$R_4 = 0$$

Hence reaction forces

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \begin{Bmatrix} -300 \\ -300 \\ -200 \\ 0 \end{Bmatrix} \text{ lb} \quad \text{Ans}$$

Element displacement (i.e. displacement in local coordinate direction)

We have already derived, $\{u'\} = [T]\{u\}$

For element 1

$$\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_5 \\ u_6 \end{Bmatrix}$$

or,

$$\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} = \begin{bmatrix} 0.707 & 0.707 & 0 & 0 \\ 0 & 0 & 0.707 & 0.707 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 5.33*10^{-4} \\ 1.731*10^{-3} \end{Bmatrix}$$

$$\begin{Bmatrix} u'_1 \\ u'_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1.6*10^{-3} \end{Bmatrix} \text{ inch}$$

For element 2

$$\begin{Bmatrix} u'_2 \\ u'_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 5.33*10^{-4} \\ 1.0731*10^{-3} \end{Bmatrix}$$

$$\begin{Bmatrix} u'_2 \\ u'_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1.073*10^{-3} \end{Bmatrix} \text{ inch}$$

Here, local displacement is equal to global displacement because in element ② local axes and global axes coincide

Element stresses :

For element 1

$$\sigma^1 = \frac{E}{le}\begin{bmatrix}-l & -m & l & m\end{bmatrix}\{u\} = \frac{E}{le}\begin{bmatrix}-l & -m & l & m\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_5\\u_6\end{Bmatrix}$$

$$= \frac{10*10^6}{56.57}\begin{bmatrix}-0.707 & -0.707 & 0.707 & 0.707\end{bmatrix}\begin{Bmatrix}0\\0\\5.33*10^{-4}\\1.073*10^{-3}\end{Bmatrix}$$

$$= 282.82 \approx 283 \text{ lb/in}^2$$

For element 2

$$\sigma^2 = \frac{10*10^6}{40}\begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}\begin{Bmatrix}0\\0\\5.33*10^{-4}\\1.073*10^{-3}\end{Bmatrix}$$

$$= 133.25 \text{ lb/in}^2$$

Member forces (element forces)

$A*\sigma = f^1$ ~~*A~~

f = element force

A = c/s area

For element 1

~~σ^1~~ $f^1 = \sigma^1 * A^1$

$= 282.82 * 1.5 = 424$ lb

+ve sign indicates tension

Here, local displacement is equal to global displacement because in element ② local axes and global axes coincide.

Element stresses :

For element 1

$$\sigma^1 = \frac{E}{le}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\{u\} = \frac{E}{le}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_5 \\ u_6 \end{Bmatrix}$$

$$= \frac{10*10^6}{56.57}\begin{bmatrix} -0.707 & -0.707 & 0.707 & 0.707 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 5.33*10^{-4} \\ 1.073*10^{-3} \end{Bmatrix}$$

$$= 282.82 \approx 283 \text{ lb/in}^2$$

For element 2

$$\sigma^2 = \frac{10*10^6}{40}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 5.33*10^{-4} \\ 1.073*10^{-3} \end{Bmatrix}$$

$$= 133.25 \text{ lb/in}^2$$

Member forces (element forces)

$A * \sigma = f^1$ ~~$*A$~~

f = element force

A = c/s area

For element 1

$f^1 = \sigma^1 * A^1$

$= 282.82 * 1.5 = 424$ lb

+ve sign indicates tension

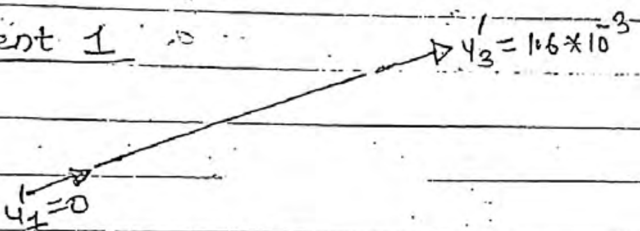
For element 2

$f^2 = \sigma^2 * A^2 = 133.25 * 1.5 \approx 200\ lb$

OR

Element forces can be calculated by treating each element as bar element.

For element 1



We have, $[K'][u'] = \{f'\}$

$[K^1]\{u'_1\} = \{f\}$

$$\frac{AE}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}0\\ 1.6*10^{-3}\end{bmatrix} = \begin{Bmatrix}f_1\\ f_3\end{Bmatrix}$$

~~$\frac{1.5*10*10^6}{56.57}\begin{bmatrix}0\\ 1.6*10^{-3}\end{bmatrix}$~~

$$\frac{1.5*10*10^6}{56.57}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ 1.6*10^{-3}\end{Bmatrix} = \begin{Bmatrix}f_1\\ f_3\end{Bmatrix}$$

$$\begin{Bmatrix}f_1\\ f_3\end{Bmatrix} = \begin{Bmatrix}-424\\ 424\end{Bmatrix}$$ which indicates tension.

lly; Do for element 2

A free body diagram of truss with reaction, applied loads & member forces as shown.

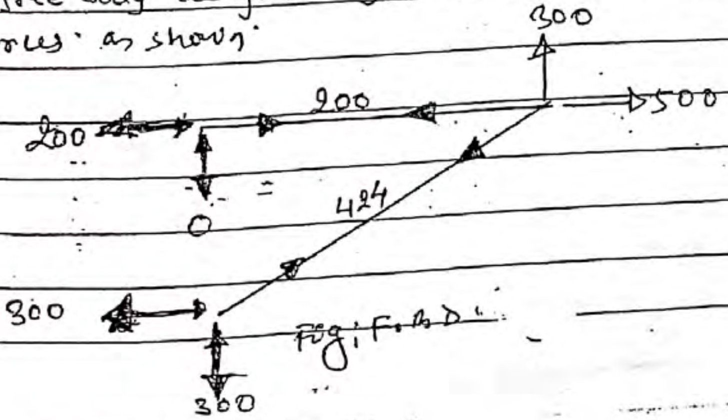


Fig: F.B.D.

Check for each joint & verify your result.

(SAP र STAAD को यही method follow गर्छ।)

example 1. (Truss)

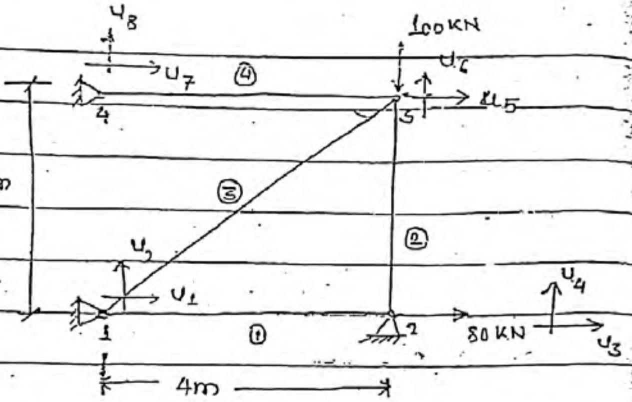
Find :-

All member forces

All support reaction

Given E = 2×10^5 N/mm^2

A = 800 mm^2



1) co-ordinate table :-

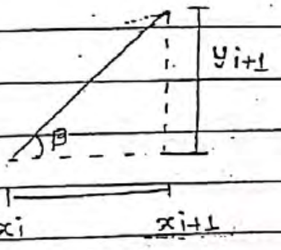
Table	x	y
1	0	0
2	4	0
3	4	3
4	0	43.

2) connectivity table :-

Element	1st node	2nd node
①	1	2
②	2	3
③	1	3
④	43	34

3) Direction cosine table

Element	le	l	m
①	4	1	0
②	3	0	1
③	5	0·8	0·6
④	4	-1	0



$$\cos\beta = l = \frac{x_{i+1} - x_i}{le}$$

$$\sin\beta = \frac{y_{i+1} - y_i}{le}$$

4) calculation of elemental stiffness matrix

$$[K_1] = \frac{A_1E_1}{L_1}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & m^2 & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$= 10^3 \begin{array}{c|cccc|c} & ① & ② & ③ & ④ & \\ \hline & 40 & 0 & -40 & 0 & ① \\ & 0 & 0 & 0 & 0 & ② \\ & -40 & 0 & 40 & 0 & ③ \\ & 0 & 0 & 0 & 0 & ④ \end{array}$$

Illy

$$K_2 = 10^3 \begin{array}{c|cccc|c} & ③ & ④ & ⑤ & ⑥ & \\ \hline & 0 & 0 & 0 & 0 & ③ \\ & 0 & 53.33 & 0 & -53.33 & ④ \\ & 0 & 0 & 40 & 0 & ⑤ \\ & 0 & -53.33 & 0 & 53.33 & ⑥ \end{array}$$

Illy

$$K_3 = 10^3 \begin{array}{c|cccc|c} & ① & ② & ⑤ & ⑥ & \\ \hline & 20.48 & 15.36 & -20.48 & -15.36 & ① \\ & 15.36 & 11.52 & -15.36 & -11.52 & ② \\ & -20.48 & -15.36 & 20.48 & 15.36 & ⑤ \\ & -15.36 & -11.52 & 15.36 & 11.52 & ⑥ \end{array}$$

l^2, lm, m^2 का value निकालें

$l^2 = \cancel{0.6^2}\ 0.8^2 \times \frac{2\times10^5 \times 800}{5 \times \cancel{10^{-6}}\ 10^6}$ = 20.

$lm = 15.36$

$m^2 = 11.52$

Illy

$$K_4 = \begin{array}{c|cccc|c} & ⑤ & ⑥ & ⑦ & ⑧ & \\ \hline & 40 & 0 & -40 & 0 & ⑤ \\ & 0 & 0 & 40 & 0 & ⑥ \\ & -40 & 0 & 40 & 0 & ⑦ \\ & 0 & 0 & 0 & 0 & ⑧ \end{array}$$

(5) Global stiffness matrix.

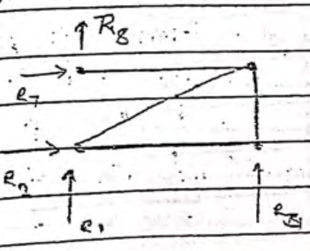
$$[K] = 10^3 \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\left[\begin{array}{cccccccc|c} ① & ② & ③ & ④ & ⑤ & ⑥ & ⑦ & ⑧ & \\ 60.48 & 15.36 & -40 & 0 & -20.48 & -15.36 & 0 & 0 & ① \\ 15.36 & 11.52 & 0 & 0 & -15.36 & -11.52 & 0 & 0 & ② \\ -40 & 0 & 40 & 0 & 0 & 0 & 0 & 0 & ③ \\ 0 & 0 & 0 & 53.33 & 0 & -53.33 & 0 & 0 & ④ \\ -20.48 & -15.36 & 0 & 0 & 60.48 & 15.36 & -40 & 0 & ⑤ \\ -15.36 & -11.52 & 0 & -53.33 & 8.26 & 64.85 & 0 & 0 & ⑥ \\ 0 & 0 & 0 & 0 & -40 & 0 & 40 & 0 & ⑦ \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ⑧ \end{array}\right]_{8\times 8}$$

Apply boundary condition and make reduced stiffness matrix.

$u_1 = 0$; $u_2 = 0$, $u_4 = 0$; $u_7 = 0$, $u_8 = 0$.

∴ reduced stiffness matrix is

$$[K^*] = 10^3 \left[\begin{array}{ccc|c} ③ & ⑤ & ⑥ & \\ -40 & 0 & 0 & ③ \\ 40 & 60.48 & 15.36 & ⑤ \\ 0 & 15.36 & 64.85 & ⑥ \end{array}\right]_{3\times 3}$$



$$\{F\} = 10^3 \begin{Bmatrix} 0+R_1 \\ 0+R_2 \\ 80+0 \\ 0+R_4 \\ 0+0 \\ -100+0 \\ 0+R_7 \\ 0+R_8 \end{Bmatrix} \begin{matrix} ① \\ ② \\ ③ \\ ④ \\ ⑤ \\ ⑥ \\ ⑦ \\ ⑧ \end{matrix}$$

∴ Reduced force matrix

$$\{F^*\} = 10^3 \begin{Bmatrix} 80+0 \\ 0+0 \\ -100+0 \end{Bmatrix}$$

$$\therefore [K^*] \begin{Bmatrix} u_3 \\ u_5 \\ u_6 \end{Bmatrix} = 10^3 \begin{Bmatrix} 80 \\ 0 \\ -100 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} u_2 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 0.4167 \\ -1.6407 \end{Bmatrix}$$

Calc^n of member force

$$\sigma' = \frac{E}{L} [-l \; -m \quad l \; m] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$= \frac{2 \times 10^5}{4000} [-1 \; 0 \; 1 \; 0] \begin{Bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{Bmatrix}$$

$$= 100 \text{ N/mm}^2$$

$$MF_1 = \sigma_1 \times A_1$$

$$= 100 \times 800$$

$$= 80 \text{KN} \; (T)$$

$$\sigma^2 = \frac{E}{L}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

$$= \frac{2\times10^5}{3000}\begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 2 \\ 0 \\ 0.4167 \\ -1.6407 \end{Bmatrix}$$

$$= \frac{-1.6407\times 2\times10^5}{3000} = -109.38 \text{ KN/mm}^2$$

$MF_2 = \sigma_2 * A_2$

$= -109.38 \times 800$

$= -87.504$ KN (C)

lly. for MF_3 and MF_4 $MF_3 = 20.83$ (C) $MF_4 = 60.67$ (T)

<u>calc^n of Rx^n</u>

$K_{11}\times u_1 + K_{12}\times u_2 + K_{13}u_3 + K_{14}\times u_4 + K_{15}\times u_5 + K_{16}\times u_6 + K_{17}\times u_7 + K_{18}\times u_8 = 0 + R_1$

$0 + 0 + (-40\times2) + 0 + (-20.48\times0.416) + (-15.36\times-1.6407) + 0 + 0 \pm 0 + R_1$

or, $\boxed{-63.31 = R_1}$

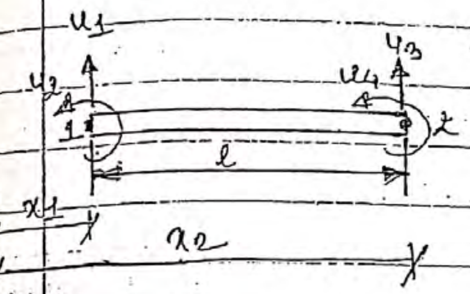
lly, $R_2 = 12.5$ KN

$R_3 = 87.5$ KN

$R_4 = -16.67$ KN

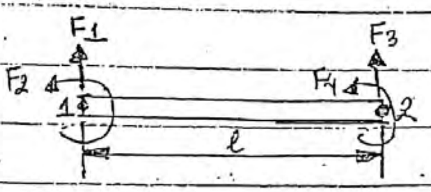
$R_5 = 0$

tation) of beam and end point deflection.



(a)

fig(a) Beam element nodal displacements
primary variables
(generalized displacement)



(b)

fig(b) generalized forces
(secondary variable)

Where

u_1 = vertical displacement of node 1.

u_2 = rotation of node 1

u_3 and u_4 same thing for node 2.

F_1 = shear force at node 1

F_2 = bending moment at node 1

F_3 and F_4 same thing for node 2.

~~The~~

The set $\{u_1, u_2, u_3, u_4\}$ is often referred to as generalized displacements and set $\{F_1, F_2, F_3, F_4)$ as generalized forces.

low, let,

load vector $\{F\} = [F_1 \quad F_2 \quad F_3 \quad F_4]^T$

displacement vector $\{u\} = [u_1 \quad u_2 \quad u_3 \quad u_4]^T$

The displacement function $u(x)$ is to be discretized such that $u(x) = f(u_1, u_2, u_3, u_4, x)$ subject to the boundary condition,

u at $x = x_1 = u_1$

$$\left.\frac{du}{dx}\right|_{x=x_1} = u_2 \text{ (at } x_1)$$

u at $x = x_2 = u_3$

$$\left.\frac{du}{dx}\right|_{x=x_2} = u_4 \ (\text{or } \theta_2)$$

Before preceeding, we assume global coordinate system, is chosen such that coordinate of node 1 is x_1 and that of node 2 is x_2.

Since there are a total four degree of freedom in an element (two per node), a four-parameter polynomial must be selected as displacement function. ie.

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \text{ ———— (i)}$$

Application of boundary condition at node, we get

$$u(x=x_1) = u_1 = a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3$$

$$\left.\frac{du}{dx}\right|_{x=x_1} = u_2 = a_1 + 2a_2 x_1 + 3a_3 x_1^2$$

$$u(x=x_2) = u_3 = a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3$$

$$\left.\frac{du}{dx}\right|_{x=x_2} = u_4 = a_1 + 2a_2 x_2 + 3a_3 x_2^2$$

Writting in matrix form

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_2 & 3x_2^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Inverting this matrix equation to findout (a_0, a_1, a_2, a_3) in terms of u_1, u_2, u_3 and u_4 and substituting result in ①, we obtain

---

Where

$H_1(x) =$

$H_2(x) =$ (

$H_3(x) =$

$H_4(x) =$

If we

$H_1(x) =$

$H_2(x) = x$

$H_3(x) = 3$

$H_4(x) =$

Note that

derived

the nodes

Hermite

(iii) or (i

(or cub

$$u_1 H_1(x) + u_2 H_2(x) + u_3 H_3(x) + u_4 H_4(x)$$

$$= \sum_{j=1}^{4} u_j H_j \longleftarrow \text{(ii)}$$

Where

$$\left.\begin{aligned} H_1(x) &= 1 - 3\left(\frac{x - x_1}{l}\right)^2 + 2\left(\frac{x - x_1}{l}\right)^3 \\ H_2(x) &= (x - x_1)\left(1 - \frac{x - x_1}{l}\right)^2 \\ H_3(x) &= 3\left(\frac{x - x_1}{l}\right)^2 - 2\left(\frac{x - x_1}{l}\right)^3 \\ H_4(x) &= (x - x_1)\left[\left(\frac{x - x_1}{l}\right)^2 - \frac{x - x_1}{l}\right] \end{aligned}\right\} \text{(iii)}$$

If we choose local coordinate system, i.e. $x_1 = 0$, $x_2 = l$, then

$$\left.\begin{aligned} H_1(x) &= 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3} \\ H_2(x) &= x - \frac{2x^2}{l} + \frac{x^3}{l^2} \\ H_3(x) &= \frac{3x^2}{l^2} - \frac{2x^3}{l^3} \\ H_4(x) &= \frac{x^3}{l^2} - \frac{x^2}{l} \end{aligned}\right\} \text{(iv)}$$

Note that the cubic interpolation function (iii) or (iv) are derived by interpolating u as well as its derivative at the nodes. Such polynomial function are known as Hermite family of interpolation function and $H_i(x)$ in (iii) or (iv) are called the Hermite cubic interpolation (or cubic spline) functions or shape function.

SYSTEM

Plots of Hermite cubic ~~function~~ interpolation function or shape function are shown in fig.

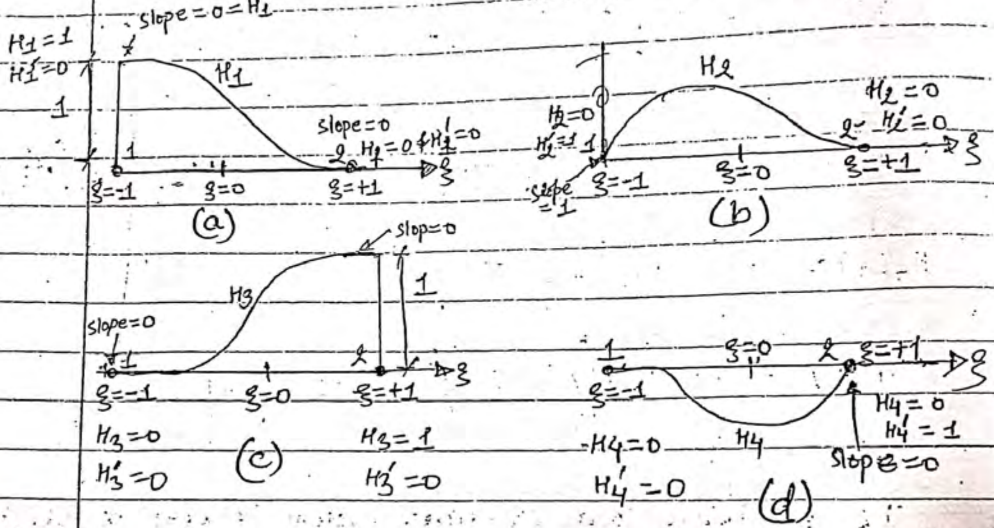


Fig:- Hermite interpolation function or shape function

Each of the shape function is of cubic order represented by:

$$H_i = a_i + b_i\xi + c_i\xi^2 + d_i\xi^3 \quad \text{—— (i)}$$

Where $i = 1, 2, 3, 4$.

~~Now, From fig (a)~~ $H_i' = b_i + 2c_i\xi + 3d_i\xi^2$ — (ii)

Now, From fig (a)

at $\xi = -1$, $H_1 = 1$ and $H_1' = 0$.

~~[illegible]~~ $\therefore H_1 = a_1 + b_1\xi + c_1\xi^2 + d_1\xi^3$

$$1 = a_1 - b_1 + c_1 - d_1 \quad \text{—— (iv)}$$

& ~~at $\xi = 1$ $H_1 = 0$, H_1~~

& $H_1' = b_1 + 2c_1\xi + 3d_1\xi^2$

$$0 = b_1 - 2c_1 + 3d_1 \quad \text{(v)}$$

$\therefore H_1 = 0$

and $H_1' = 0$

On solving

$a_1 = \frac{1}{2}$

$\therefore \boxed{H_1 = \frac{1}{2}}$

Similarly fr

~~At ξ =~~

At $\xi = -1$,

$\therefore$ H

&

At $\xi = +1$

$H_2 =$

H_2'

On solvin

$H_2 =$

Similarly

we get

$H_3 =$

& $H_4 =$

These ar

of nat

i.e. $H_2 =$

usly deri

and

On solving ⓘⓥ ⓥ & (vii), (vii)

$a_1 = \frac{1}{2}$, $b_1 = -\frac{3}{4}$, $c_1 = 0$ & $d_1 = 1/4$

$$\therefore \boxed{H_1 = \frac{1}{2} - \frac{3}{4}\xi + 0 * \xi^2 + \frac{1}{4}\xi^3 = \frac{1}{4}\left(2 - 3\xi + \xi^3\right)}$$

Similarly from fig (b)

At ~~ξ =~~ $H_2 = a_2 + b_2\xi + c_2\xi^2 + d_2\xi^3$

$H_2' = b_2 + 2c_2\xi + 3d_2\xi^2$

At $\xi = -1$, $H_2 = 0$ & $H_2' = 1$

$\therefore H_2 = 0 = a_2 - b_2 + c_2 - d_2$

& $H_2' = 1 = b_2 - 2c_2 + 3d_2$

At $\xi = +1$ - $H_2 = 0$ & $H_2' = 0$

$H_2 = 0 = a_2 + b_2 + c_2 + d_2$

$H_2' = 0 = b_2 + 2c_2 + 3d_2$

On solving

$$H_2 = \frac{1}{4}\left(1 - \xi - \xi^2 + \xi^3\right)$$

Similarly solve for H_3 and H_4 from fig ⓒ & ⓓ, we get

$$H_3 = \frac{1}{4}\left(2 + 3\xi - \xi^3\right)$$

$$\& \quad H_4 = \frac{1}{4}\left(-1 - \xi + \xi^2 + \xi^3\right)$$

These are the hermite shape function in terms of natural coordinate system, or put $x_1 = -1$ and & $x_2 = +1$ and $x_2 - x_1 = l = 2$ in the eqn (iii) previously derived eqn, we get Same result.

We have already derived that

$u(x) = u_1 H_1(x) + u_2 H_2(x) + u_3 H_3(x) + u_4 H_4(x)$ in global coordinate system

Now, in natural coordinate system,

$$u(\xi) = H_1u_1 + H_2u_2 + H_3u_3 + H_4u_4$$

$$\Rightarrow u(\xi) = H_1u_1 + H_2\left(\frac{du}{d\xi}\right)_1 + H_3u_3 + H_4\left(\frac{du}{d\xi}\right)_2 \quad \text{— (i)}$$

We have, $\xi = \frac{2(x-x_1)}{x_2-x_1} - 1$

$$\Rightarrow \frac{d\xi}{dx} = \frac{2}{x_2-x_1} = \frac{2}{l}$$

and, $\frac{du}{dx} = \frac{du}{d\xi} \times \frac{d\xi}{dx}$

$$\Rightarrow \frac{du}{dx} = \frac{du}{d\xi} \times \frac{2}{l} \qquad \therefore \frac{du}{d\xi} = \frac{l}{2}\frac{du}{dx} \quad \text{— (ii)}$$

Hence, now from <i>

$$u(\xi) = H_1u_1 + \frac{l}{2}H_2\left(\frac{du}{dx}\right)_1 + H_3u_3 + \frac{l}{2}H_4\left(\frac{du}{dx}\right)_2$$

$$\Rightarrow u(\xi) = H_1u_1 + \frac{l}{2}H_2u_2 + H_3u_3 + \frac{l}{2}H_4u_4$$

In matrix form,

$$\{u\} = [H]\{\underline{u}\}$$

where

$$[H] = \begin{bmatrix} H_1 & \frac{l}{2}H_2 & H_3 & \frac{l}{2}H_4 \end{bmatrix}$$

$$\underline{u} = \text{nodal displacement} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$V_e = \frac{EI}{2}$

$\therefore \frac{M}{EI} = \frac{d}{}$

We know, Fro

and

$\Rightarrow \frac{d^2u}{dx}$

put

$\left(\frac{d^2u}{dx^2}\right)^2 = \left\{\frac{4}{l}\right.$

$= \frac{16}{l^4}$

Since nodal

be sepe

$\int_l \left(\frac{d^2u}{dx^2}\right) dx$ ——— ⟨III⟩

$\because \boxed{\frac{M}{EI} = \frac{d^2u}{dx^2}}$ & $\boxed{U_e = \int_l \frac{M^2}{2EI} dx} = \int_l \frac{\left(EI \frac{d^2u}{dx^2}\right)^2}{2EI} dx$

$$= \frac{EI}{2} \int_l \left(\frac{d^2u}{dx^2}\right)^2 dx$$

~~We know~~, From above, $\frac{du}{dx} = \frac{2}{l} \frac{du}{d\xi}$

and $\frac{d^2u}{dx^2} = \frac{d}{dx}\left(\frac{2}{l} \frac{du}{d\xi}\right) = \frac{d}{d\xi}\left(\frac{2}{l} \frac{du}{dx}\right)$

$\Rightarrow \frac{d^2u}{dx^2} = \frac{d}{d\xi}\left(\frac{2}{l} * \frac{2}{l} \frac{du}{d\xi}\right) = \frac{4}{l^2} \frac{d^2u}{d\xi^2}$

put, $\{u\} = [H]\{\underline{u}\}$

$$\left(\frac{d^2u}{dx^2}\right)^2 = \left\{\frac{4}{l^2} \frac{d^2}{d\xi^2} [H]\{\underline{u}\}\right\}^2$$

$$= \frac{16}{l^4} \left\{\frac{d^2}{d\xi^2} [H]\{\underline{u}\}\right\}^2$$

Since nodal displacement i.e. $\{\underline{u}\}$ constant term which can be seperated from differentiation

$$\left(\frac{d^2u}{dx^2}\right) \quad \frac{}{l^4}\left(\frac{}{d\xi^2}\right)$$

$$\left(\frac{d^2u}{dx^2}\right)^2 = \frac{16}{l^4}\left[\frac{d^2[H]}{d\xi^2}\right]^T\left[\frac{d^2[H]}{d\xi^2}\right]\{\underline{u}\}^T\{\underline{u}\}$$

$$\because \quad [A]^2 = [A]^T[A]$$

$$\left(\frac{d^2u}{dx^2}\right)^2 = \frac{16}{l^4}\{\underline{u}\}^T\left[\frac{d^2[H]}{d\xi^2}\right]^T\left[\frac{d^2[H]}{d\xi^2}\right]\{\underline{u}\}$$

Substituting this on ⟨iii⟩ and $\frac{d\xi}{dx} = 2/l$ or, $dx = \frac{l}{2}d\xi$ on ⟨iii⟩ we get

$$u_e = \frac{EI}{2}\int_l\left(\frac{d^2u}{dx}\right)^2 dx = \frac{EI}{2}\int_{-1}^{+1}\frac{16}{l^4}\{\underline{u}\}^T\left[\frac{d^2[H]}{d\xi^2}\right]^T\left[\frac{d^2[H]}{d\xi^2}\right]\{\underline{u}\}\frac{l}{2}d\xi$$

Which is in the form of $V_e = \frac{1}{2}\underline{u}^T K \underline{u}$

on comparing

$$[K] = EI\int_{-1}^{+1}\frac{16}{l^4}\left[\frac{d^2[H]}{d\xi^2}\right]^T\left[\frac{d^2[H]}{d\xi^2}\right] l/2 \, d\xi$$

$$= \frac{8EI}{l^3}\int_{-1}^{+1}\left[\frac{d^2[H]}{d\xi^2}\right]^T\left[\frac{d^2[H]}{d\xi^2}\right] d\xi \quad \text{— ⟨iv⟩}$$

Now, $[H] = \left[H_1 \quad l/_2 H_2 \quad H_3 \quad l/_2 H_4\right]$

$$\frac{d^2}{d\xi^2}[H] = \frac{d^2}{d\xi^2}\left\{\left[H_1 \quad l/_2 H_2 \quad H_3 \quad l/_2 H_4\right]\right\}$$

$$\Rightarrow \frac{d^2}{d\xi^2}(H_1) = \frac{d}{d\xi^2}\left(\frac{1}{4}\left(2 - 3\xi + \xi^3\right)\right) = \frac{3\xi}{2}$$

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$$\frac{d^2(H_3)}{d\xi^2} = \frac{d^2}{d\xi^2}\left(\frac{1}{4}(2+3\xi-\xi^3)\right) = \frac{-3}{2}\xi$$

$$\frac{d^2(H_4)}{d\xi^2} = \frac{d^2}{d\xi^2}\left(\frac{1}{4}(-1-\xi+\xi^2+\xi^3)\right) = \frac{1+3\xi}{2}$$

$$\therefore \left\{\frac{d^2}{d\xi^2}[H]\right\} = \begin{bmatrix} \frac{3}{2}\xi & \frac{-1+3\xi}{2}\frac{l}{2} & -\frac{3}{2}\xi & \frac{1+3\xi}{2}\, l/2 \end{bmatrix}$$

$$\& \left[\frac{d^2}{d\xi^2}[H]\right]^T = \begin{bmatrix} \frac{3}{2}\xi \\ \frac{-1+3\xi}{2}\, l/2 \\ -\frac{3}{2}\xi \\ \frac{1+3\xi}{2}\, l/2 \end{bmatrix}$$

Substituting these in $\langle Ev \rangle$

$$] = \frac{8EI}{l^3}\int_{-1}^{+1} \begin{bmatrix} \frac{3}{2}\xi \\ \frac{-1+3\xi}{2}\, l/2 \\ -\frac{3}{2}\xi \\ \frac{1+3\xi}{2}\, l/2 \end{bmatrix} \begin{bmatrix} \frac{3}{2}\xi & \frac{-1+3\xi}{2}\, l/2 & -\frac{3}{2}\xi & \frac{1+3\xi}{2}\, l/2 \end{bmatrix} d\xi$$

$$= \frac{8EI}{l^3}\int_{-1}^{+1} \begin{bmatrix} \frac{9}{4}\xi^2 & \frac{3}{8}\xi(-1+3\xi) & -\frac{9}{4}\xi^2 & \frac{3\xi(1+3\xi)l}{8} \\ & \left(\frac{-1+3\xi}{4}\right)^2 l^2 & -\frac{3}{8}\xi(-1+3\xi)l & \frac{-1+9\xi^2}{16}l^2 \\ & \text{Symmetric} & \frac{9}{4}\xi^2 & -\frac{3}{8}\xi(1+3\xi)l \\ & & & \left(\frac{1+3\xi}{4}\right)^2 l^2 \end{bmatrix} d\xi$$

$$\underset{\text{even function}}{\int_{-1}^{+1} \xi^2 d\xi = \frac{2}{3}} \quad \underset{\text{odd function}}{\int_{-1}^{+1} \xi \, d\xi = 0} \quad \int_{-1}^{+1} d\xi = 2$$

on integrating and arranging

$$k = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

Which is symmetric.

ELEMENT LOAD VECTOR

Consider, uniformly distributed load of w in the element. The usual approach is to replace the distributed load with nodal forces and moments such that mechanical work done by nodal load system is equivalent to that done by the distributed load.

Now, Mechanical work done performed by ~~distr~~ uniformly distributed load can be expressed as

$$W = \int_0^L w\{u\}\,dx = \int_0^L w\,[N]\{\underline{u}\}\,dx$$

$$= w\int_{-1}^{+1} [N]\{\underline{u}\}\,\frac{l}{2}\,d\xi \qquad \left(\because \frac{d\xi}{dx} = \frac{2}{l}\right)$$

Here $\{\underline{u}\}$ is nodal displacement matrix which is constant.

$$\left[2 \int_{-1} [H]\, d\xi \right]^T \{u\} \quad — ①$$

$$\Rightarrow W = \{F_e\}^T \{u\}$$ [This is the work done by nodal forces] — ②

Where $\{F_e\}$ = Nodal force matrix.

From ① & ②

$$\{Fe\} = \frac{wl}{2} \int_{-1}^{+1} [H]\, d\xi$$

$$= \frac{wl}{2} \int_{-1}^{+1} \left[H_1 \quad \frac{l}{2} H_2 \quad H_3 \quad l/_2 H_4 \right] d\xi$$

$$= \frac{wl}{2} \int_{-1}^{1} \left[\frac{1}{4}(2 - 3\xi + \xi^3) \quad \frac{1}{4}\frac{l}{2}(1 - \xi - \xi^2 + \xi^3) \quad \frac{1}{4}(2 + 3\xi - \xi^3) \quad \frac{1}{4}\frac{l}{2}(-1 - \xi + \xi^2 + \xi^3) \right] d\xi$$

$$= \frac{wl}{2} \left[1 \quad \frac{l}{6} \quad 1 \quad -\frac{l}{6} \right] = \left[\frac{wl}{2} \quad \frac{wl^2}{12} \quad \frac{wl}{2} \quad \frac{-wl^2}{12} \right]$$

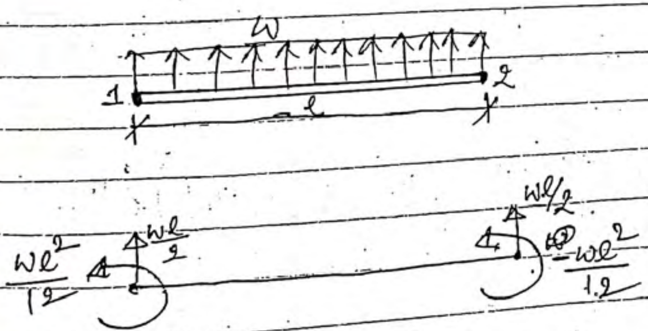

Fig :- The equivalance of a uniformly distributed load to the corresponding nodal forces.

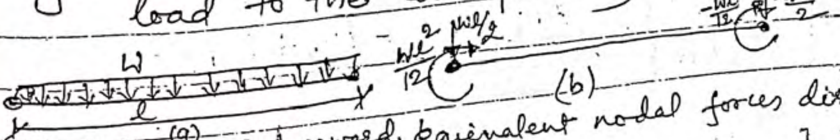

Fig :- If loading system is downward, equivalent nodal forces diz]

Bending moment and Shear

We know, $M = EI\frac{d^2u}{dx^2}$

We have already derived $\frac{d^2u}{dx^2} = \frac{4}{\ell^2}\frac{d^2u}{d\xi^2}$

$$\therefore M = EI\cdot\frac{4}{\ell^2}\frac{d^2u}{d\xi^2} = \frac{4EI}{\ell^2}\frac{d^2}{d\xi^2}\left([H]\{\underline{u}\}\right)$$

$$= \frac{4EI}{\ell^2}\frac{d^2}{d\xi^2}\left(\left[H_1 \quad \frac{\ell}{2}H_2 \quad H_3 \ \cancel{B} \ \frac{\ell}{2}H_4\right]\right)\{u_1 \quad u_2 \quad u_3 \quad u_4\}$$

$$= \frac{4EI}{\ell^2}\left[\frac{6\xi}{4} \quad \frac{\ell}{2}\left(-2+\frac{6\xi}{4}\right) \quad \frac{-6\xi}{4} \quad \frac{\ell}{2}\left(2+\frac{6\xi}{4}\right)\right]\{u_1 u_2 u_3 u_4\}$$

$$\boxed{M = \frac{EI}{\ell^2}\left[6\xi u_1 + (3\xi-1)\ell u_2 - 6\xi u_3 + (3\xi+1)\ell u_4\right]}$$

Shear force $V = \frac{dM}{dx} = \frac{d}{dx}\left[\frac{4EI}{\ell^2}\frac{d^2u}{d\xi^2}\right]$

$$= \frac{d^2}{d\xi^2}\left[\frac{4EI}{\ell^2}\frac{du}{dx}\right] = \frac{d^2}{d\xi^2}\left[\frac{4EI}{\ell^2} * \frac{2}{\ell}\frac{du}{d\xi}\right]$$

$$V = \frac{8EI}{\ell^3}\frac{d^3u}{d\xi^3} = \frac{8EI}{\ell^3}\frac{d^3}{d\xi^3}[H]\{\underline{u}\}$$

$$V = \frac{8EI}{\ell^3}\frac{d^3}{d\xi^3}\left(\left[H_1 \quad \frac{\ell}{2}H_2 \quad H_3 \quad \frac{\ell}{2}H_4\right]\right)\{u_1 \quad u_2 \quad u_3 \quad u_4\}$$

$$V = \frac{8EI}{\ell^3}\left[\frac{3}{2} \quad \frac{3\ell}{4} \quad \frac{-3}{2} \quad \frac{3\ell}{4}\right]\{u_1 \quad u_2 \quad u_3 \quad u_4\}$$

$$\boxed{V = \frac{6EI}{\ell^3}\left[2u_1 + \ell u_2 - 2u_3 + \ell u_4\right]}$$

① The simply
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Solution

Here, we h
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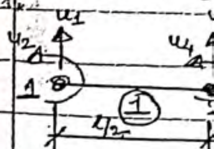


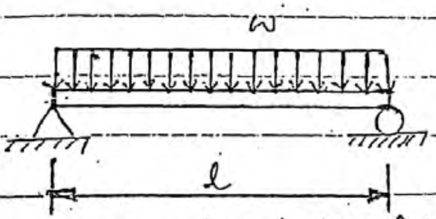
Fig(b) Noda
nodal

$\frac{w\left(\frac{\ell}{2}\right)^2}{12} = \frac{w\ell^2}{48}$

$\frac{w\left(\frac{\ell}{2}\right)}{2} = \frac{w\ell}{4}$

Step 1 Stiffne

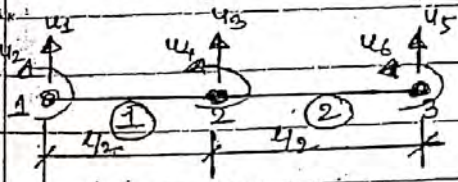
length elements and work-equivalent nodal loads obtain a finite element solution for the deflection at mid span and compare it to the solution given by elementary beam theory.



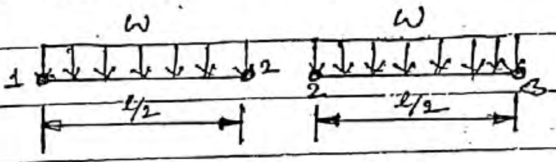
Fig(a)- uniformly loaded beam.

Solution

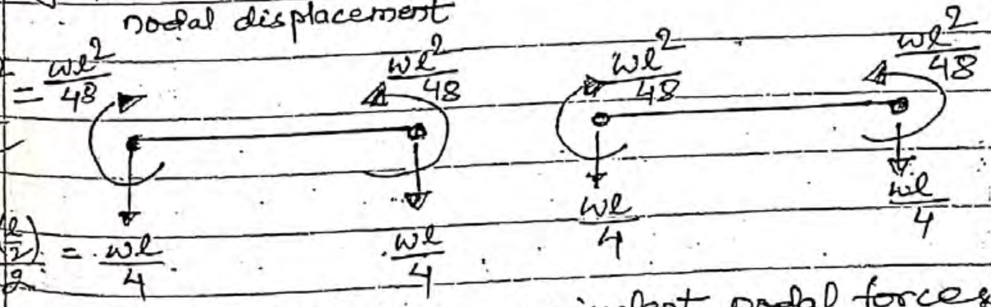
Here, we have to compute deflection at mid point, so we have to consider one node at mid point. Hence, the given fig. will be of two equal length elements.



Fig(b) Nodal, element and nodal displacement



Fig(c) Element loading.



Fig(c) Work-equivalent nodal forces.

p1 Stiffness matrix Here, element ① & element ② are same

p.t.o

$$\left(\frac{l}{2}\right) \begin{array}{cccc|c} \frac{6l}{2} & \frac{4l^2}{4} & -\frac{6l}{2} & \frac{2l^2}{4} & 2 \\ -12 & -\frac{6l}{2} & 12 & -\frac{6l}{2} & 3 \\ \frac{6l}{2} & \frac{2l^2}{4} & -\frac{6l}{2} & \frac{4l^2}{4} & 4 \end{array}$$

For ~~element 2~~

$$[K^I] = \frac{8EI}{l^3} \begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ 12 & 3l & -12 & 3l & 1 \\ 3l & l^2 & -3l & \frac{l^2}{2} & 2 \\ -12 & -3l & 12 & -3l & 3 \\ 3l & \frac{l^2}{2} & -3l & l^2 & 4 \end{array}$$

$$[K^2] = \frac{8EI}{l^3} \begin{array}{cccc|c} 3 & 4 & 5 & 6 & \\ 12 & 3l & -12 & 3l & 3 \\ 3l & l^2 & -3l & \frac{l^2}{2} & 4 \\ -12 & -3l & 12 & -3l & 5 \\ 3l & \frac{l^2}{2} & -3l & l^2 & 6 \end{array}$$

Step 2

Assembled global stiffness matrix (6*6)

$$[K] = \frac{8EI}{l^3} \begin{array}{cccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & \\ 12 & 3l & -12 & 3l & 0 & 0 & 1 \\ 3l & l^2 & -3l & \frac{l^2}{2} & 0 & 0 & 2 \\ -12 & -3l & 24 & 0 & -12 & 3l & 3 \\ 3l & \frac{l^2}{2} & 0 & 2l^2 & -3l & \frac{l^2}{2} & 4 \\ 0 & 0 & -12 & -3l & 12 & -3l & 5 \\ 0 & 0 & 3l & \frac{l^2}{2} & -3l & l^2 & 6 \end{array}$$

Let R_1 and R_5 the reaction at supports,

$$\begin{array}{c|c} \frac{\quad}{48} & 2 \\ -\frac{wl}{4} & 3 \\ \frac{wl^2}{48} & 4 \end{array} \qquad \begin{array}{c|c} -\frac{wl^2}{48} & 4 \\ -\frac{wl}{4} & 5 \\ \frac{wl^2}{48} & 6 \end{array}$$

On Assembling,

Element load matrix $\{F^{eq}\} = \left\{\begin{array}{c} -\frac{wl}{4} \\ -\frac{wl^2}{48} \\ -\frac{wl}{2} \\ 0 \\ -\frac{wl}{4} \\ \frac{wl^2}{48} \end{array}\right\} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$

Now, Reaction matrix.

$$R = 0 \left\{\begin{array}{c} R_1 \\ 0 \\ 0 \\ 0 \\ R_5 \\ 0 \end{array}\right\} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Global force vector vector $\{F\} = \left\{\begin{array}{c} -\frac{wl}{4} + R_1 \\ -wl^2/48 \\ -wl/2 \\ 0 \\ -\frac{wl}{4} + R_5 \\ wl^2/48 \end{array}\right\}$

Global equilibrium equation

$$\frac{8EI}{l^3}\begin{bmatrix} 12 & 3l & -12 & 3l & 0 & 0 \\ 3l & l^2 & -3l & l^2/2 & 0 & 0 \\ -12 & -3l & 24 & 0 & -12 & 3l \\ 3l & l^2/2 & 0 & 2l^2 & -3l & l^2/2 \\ 0 & 0 & -12 & -3l & 12 & -3l \\ 0 & 0 & 3l & l^2/2 & -3l & l^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} -\frac{wl}{4}+R_1 \\ -wl^2/48 \\ -wl/2 \\ 0 \\ -\frac{wl}{4}+R_5 \\ wl^2/48 \end{Bmatrix}$$

note that boundary conditions, $u_1 = u_5 = 0$

Also, slope of the beam at mid span is zero,
Since the loading and support condition are same. ie. $u_4 = 0$.

Deleting rows and column corresponding to 1, 4, 5 we obtain following reduced system.

$$\frac{8EI}{l^3}\begin{bmatrix} l^2 & -3l & 0 \\ -3l & 24 & 3l \\ 0 & 3l & l^2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} -wl^2/48 \\ -wl/2 \\ wl^2/48 \end{Bmatrix}.$$

on solving,

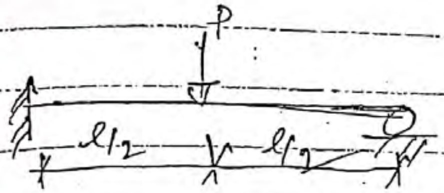
$$u_2 = \frac{-wl^3}{24EI}.$$

$$u_3 = -\frac{5wl^4}{384EI}$$

$$u_6 = \frac{wl^3}{24EI}$$

The nodal displacement results from finite element analysis of this example are exactly the results obtained strength of material.

midspan deflection.



Solution

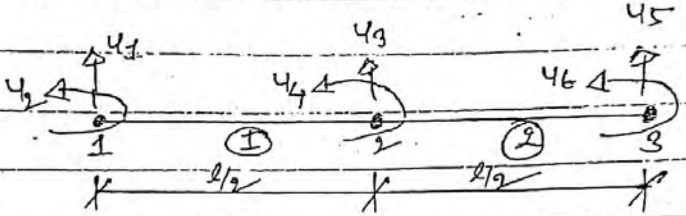


Fig 1- Element and nodal displacement designation

step 1 stiffness matrix

For element 1

$$[K^1] = [K^2] = \frac{EI}{(l/2)^3}\begin{bmatrix} 12 & 6l/2 & -12 & 6l/2 \\ 6l/2 & 4l^2/4 & -6l/2 & 2l^2/4 \\ -12 & -6l/2 & 12 & -6l/2 \\ 3l & 2l^2/4 & -6l/2 & 4l^2/4 \end{bmatrix}$$

$$= \frac{8EI}{l^3}\begin{bmatrix} 12 & 3l & -12 & 3l \\ 3l & l^2 & -3l & l^2/2 \\ -12 & -3l & 12 & -3l \\ 3l & l^2/2 & -3l & l^2 \end{bmatrix}$$

Now, Global assembled stiffness matrix as in previous problem

$$[K] = \frac{8EI}{l^3}\begin{bmatrix} 12 & 3l & -12 & 3l & 0 & 0 \\ 3l & l^2 & -3l & l^2/2 & 0 & 0 \\ -12 & -3l & 24 & 0 & -12 & 3l \\ 3l & l^2/2 & 0 & 2l^2 & -3l & l^2/2 \\ 0 & 0 & -12 & -3l & 12 & -3l \\ 0 & 0 & 3l & l^2/2 & -3l & l^2 \end{bmatrix}$$

Reaction matrix = $\begin{Bmatrix} R_1 \\ R_2 \\ 0 \\ 0 \\ R_5 \\ 0 \end{Bmatrix}$

There is no load on each element, therefore, no element load matrix.

Force matrix due to externally applied load = $\begin{Bmatrix} 0 \\ 0 \\ -P \\ 0 \\ 0 \\ 0 \end{Bmatrix}$

Global load vector = $\begin{Bmatrix} R_1 \\ R_2 \\ -P \\ 0 \\ R_5 \\ 0 \end{Bmatrix}$

using general form

$$[K]\{U\} = \{F\}$$

$$\frac{8EI}{l^3}\begin{bmatrix} 12 & 3l & -12 & 3l & 0 & 0 \\ 3l & l^2 & -3l & l^2/2 & 0 & 0 \\ -12 & -3l & 24 & 0 & -12 & 3l \\ 3l & l^2/2 & 0 & 2l^2 & -3l & l^2/2 \\ 0 & 0 & -12 & -3l & 12 & -3l \\ 0 & 0 & 3l & l^2/2 & -3l & l^2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ -P \\ 0 \\ R_5 \\ 0 \end{Bmatrix}$$

$$\frac{8EI}{\ell^3}\begin{bmatrix}24 & 0 & 3\ell\\ 0 & 2\ell^2 & \ell^2/2\\ 3\ell & \ell^2/2 & \ell^2\end{bmatrix}\begin{Bmatrix}u_3\\ u_4\\ u_6\end{Bmatrix}=\begin{Bmatrix}-P\\ 0\\ 0\end{Bmatrix}$$

$$\frac{8EI}{\ell^3}\left(24u_3+3\ell u_6\right\} = -P \quad \text{— (i)}$$

$$\frac{8EI}{\ell^3}\left(2\ell^2 u_4+\frac{\ell^2}{2}u_6\right\} = 0$$

$$2\ell^2 u_4+\frac{\ell^2}{2}u_6 = 0 \quad \text{— (ii)}$$

$$\frac{8EI}{\ell^3}\left(\ell_3 u_3+\frac{\ell^2}{2}u_4+\ell^2 u_6\right) = 0 \quad \text{— (iii)}$$

Solving these equations,

We get, $u_3 = \frac{-7P\ell^3}{768EI}$ $u_4 = \frac{-P\ell^2}{128EI}$ $u_6 = \frac{P\ell^2}{32EI}$

Calulation of reaction forces

Substitution of these nodal displacement on the Global equilibrium form (above), we get,

$$\frac{8EI}{\ell^3}\left[12u_1+3\ell u_2-12u_3+3\ell u_4+0+0\right] = R_1$$

u_1 & $u_2 = 0$,

$$\frac{8EI}{\ell^3}\left[-12\left(\frac{-7P\ell^3}{768EI}\right)+3\ell*\left(\frac{-P\ell^2}{128EI}\right)\right] = R_1$$

$$R_1 = \frac{11P}{16}$$

For R_2.

$$\frac{8EI}{\ell^3}\left[\cancel{12*0}-3\ell*0+\ell^2*0-3\ell*u_3+\frac{\ell^2}{2}u_4+0+0\right] = R_2$$

$$R_2 = \frac{3PL}{16}$$

for R_5

$$\frac{8EI}{l^3}\left[0+0-12*y_3-3l*y_4+12*0-3l*y_6\right]=R_5$$

$$R_5=\frac{5P}{16}$$

~~Checking~~

-ped into two elements. Note that nodes are positioned at the points of load application and the supports.

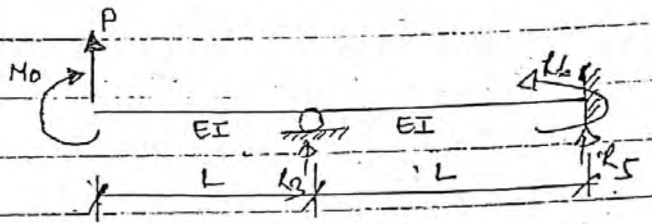


fig:- Give problem

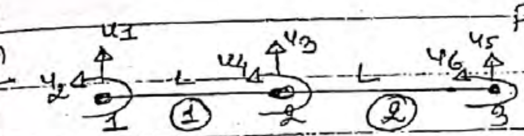


Fig:- node numbering and nodal displacement designation

Here element 1 and 2 are same,

<u>stiffness matrix</u>

So, $[K^1] = [K]^2$

$$[K^1] = \frac{EI}{L^3}\begin{array}{c|c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} & \leftarrow \text{Global DOF} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} & \begin{matrix} \downarrow \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \end{array}$$

$$[K^2] = \frac{EI}{L^3}\begin{array}{c|c} \begin{matrix} 3 & 4 & 5 & 6 \end{matrix} & \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} & \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix} \end{array}$$

On assembling, Global stiffness matrix,

p.t.o

$$[K] = \frac{EI}{L^3}\begin{array}{c|cccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline & 12 & 6L & -12 & 6L & 0 & 0 & 1 \\ & 6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 2 \\ & -12 & -6L & 24 & 0 & -12 & 6L & 3 \\ & 6L & 2L^2 & [illegible] & 8L^2 & -6L & 2L^2 & 4 \\ & 0 & 0 & -12 & -6L & 12 & -6L & 5 \\ & 0 & 0 & 6L & 2L^2 & -6L & 4L^2 & 6 \end{array}$$

∴ Now, Global load vector.

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \\ F_3 \\ M_3 \end{Bmatrix} = \begin{Bmatrix} P \\ -M_0 \\ R_2 \\ 0 \\ R_5 \\ R_6 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

Now, Global equilibrium.

$[K]\{u\}=\{F\}$

$$\frac{EI}{L^3}\begin{bmatrix} \overset{1}{12} & \overset{2}{6L} & \overset{3}{-12} & \overset{4}{6L} & \overset{5}{0} & \overset{6}{0} \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ \cancel{-12} & \cancel{-6L} & \cancel{24} & \cancel{0} & \cancel{-12} & \cancel{6L} \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ \cancel{0} & \cancel{0} & \cancel{-12} & \cancel{-6L} & \cancel{12} & \cancel{-6L} \\ \cancel{0} & \cancel{0} & \cancel{6L} & \cancel{2L^2} & \cancel{-6L} & \cancel{4L^2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} P \\ -M_0 \\ \cancel{R_2} \\ 0 \\ \cancel{R_5} \\ \cancel{R_6} \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ \cancel{3} \\ 4 \\ \cancel{5} \\ \cancel{6} \end{matrix}$$

Applying boundary condition, $u_3 = u_5 = u_6 = 0$,

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & 6L \\ 6L & 4L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} P \\ -M_0 \\ 0 \end{Bmatrix}$$

Problem No:4

A cantile

to uniform

length.

beam i

beam e

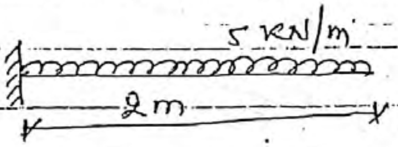
Soln

Stiffnes

$[K] =$

$=$

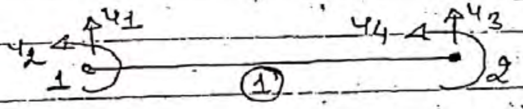
A cantilever beam ~~is subjected~~ 2 m long is subjected to uniformly distributed load of 5 KN/m over its entire length. Find the slope and deflection of cantilver beam its free end and mid point using one beam element model.



$$EI = 2 * 10^{12}\ KN\ mm^2 = 2.5 * 10^{12} * 10^{-6} * 10^{-3}\ KN/m^2$$

$$= 2.5 * 10^{3}\ KN m^2$$

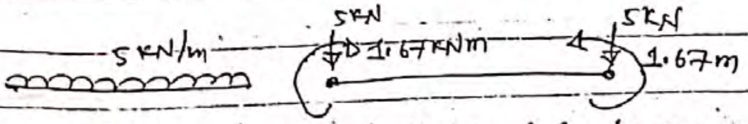
Sol^n



Stiffness matrix

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

$$= \frac{2.5*10^3}{2^3}\begin{bmatrix} 12 & 12 & -12 & 12 \\ 12 & 16 & -12 & 8 \\ -12 & -12 & 12 & -12 \\ 12 & 8 & -12 & 16 \end{bmatrix}$$



equivalent nodal forces

https://civinnovate.com/civil-engineering-notes/

$$\begin{Bmatrix} -wl^2/12 \\ -wl/12 \\ wl^2/12 \end{Bmatrix} = \begin{Bmatrix} -1.67 \\ -5 \\ 1.67 \end{Bmatrix}$$

And Global load vector

$$\{F\} = \begin{Bmatrix} -5+R_1 \\ -1.67+R_2 \\ -5 \\ 1.67 \end{Bmatrix}$$

We have,

$$[K]\{u\} = \{F\}$$

$$312500 \begin{bmatrix} 12 & 12 & -12 & 12 \\ 12 & 16 & -12 & 8 \\ -12 & -12 & 12 & -12 \\ 12 & 8 & -12 & 16 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} -5+R_1 \\ -1.67+R_2 \\ -5 \\ 1.67 \end{Bmatrix}$$

Applying boundary condition, $u_1 = u_2 = 0$, we get following reduced form.

$$312500 \begin{bmatrix} 12 & -12 \\ -12 & 16 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} -5 \\ 1.67 \end{Bmatrix}$$

on solving,

$u_3 = -3.997 * 10^{-3}$ ~~mm~~ m

$u_4 = -2.664 * 10^{-3}$ radian

Now, Deflection at mid span

we have

$$u(\xi) = H_1 u_1 + \frac{l}{2} H_2 u_2 + H_3 u_3 + \frac{l}{2} H_4 u_4$$

At mid span $\xi = 0$

$$+ \frac{2}{2}\left(\frac{-1}{4}\right)\left(-2.664*10^{-3}\right)$$

$$= -1.3325*10^{-3}.$$ Ans

b)

Fig(a) Beam with point load

Fig(b) equivalent nodal forces due to point load

Hence load vector $\{f\} = \begin{Bmatrix} -\frac{Pb}{l} \\ -\frac{Pab^2}{l^2} \\ -\frac{Pa}{l} \\ \frac{Pa^2b}{l^2} \end{Bmatrix}$

c) ~~For uniformly distrib~~

c) For uniformly derive varying load

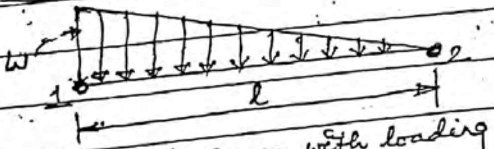


Fig:- Beam with loading

Consider a beam of length l and Subjected to linearly varying distributed load w(x).

1st we note that, the load intensity at a point x distance from node 1 is given by

$$\frac{W}{l} = \frac{W(x)}{l-x}$$

$$\Rightarrow W(x) = W\left(1-\frac{x}{l}\right)$$

Now, we have to evaluate contribution of this load to the nodes.

We have, on equating work done by given loading to the work done by nodal force, nodal force vector is given by following expression

~~$F = f$~~

$$\{F\} = \int_0^l w(x)\,N(x)\,dx$$

Integration will be easy on natural coordinate system,

$$w(x) = w\left(1 - \frac{x}{l}\right) = w\left(1 - \frac{1+\xi}{2}\right) = w\left(\frac{1-\xi}{2}\right)$$

$$\because \xi = \frac{2}{x_2 - x_1}(x - x_1) - 1$$

$$\xi = \frac{2}{l}(x-0) - 1$$

$$\frac{x}{l} = \frac{1+\xi}{l}$$

$$\therefore \frac{d\xi}{dx} = \frac{2}{l}$$

$$\therefore \{F\} = \int_{-1}^{1} w(\xi)\{N(\xi)\}^T \frac{l}{2}\,d\xi$$

$$\{F\} = \int_{-1}^{1} w\left(\frac{1-\xi}{2}\right) \begin{Bmatrix} \frac{1}{4}(2 - 3\xi + \xi^3) \\ \frac{l}{2}\,1/4\,(1 - \xi - \xi^2 + \xi^3) \\ 1/4\,(2 + 3\xi - \xi^3) \\ \frac{l}{2}\,1/4\,(-1 - \xi + \xi^2 + \xi^3) \end{Bmatrix} l/2\,d\xi$$

$$\{F\} = \frac{wl}{16}\int_{-1}^{1} (1-\xi) \begin{Bmatrix} (2 - 3\xi + \xi^3) \\ \frac{l}{2}(1 - \xi - \xi^2 + \xi^3) \\ (2 + 3\xi - \xi^3) \\ \frac{l}{2}(-1 - \xi + \xi^2 + \xi^3) \end{Bmatrix} d\xi$$

Now, $\int_{-1}^{1}(1-\xi)(2-3\xi+\xi^3)\,d\xi = \int_{-1}^{1}(2-3\xi+\xi^3-2\xi+3\xi^2-\xi^4)\,d\xi$

$\Rightarrow$ ① $= \int_{-1}^{+1}(2-5\xi+3\xi^2+\xi^3-\xi^4)\,d\xi$

And noting that,

$\int_{-1}^{1}\xi^2\,d\xi = \frac{2}{3}$ (even function) $\int_{-1}^{1}\xi\,d\xi = 0$ (odd function) $\int_{-1}^{1}d\xi = 2$

$\Rightarrow 2*2 - 0 + 3*\frac{2}{3} + 0 - \frac{2}{5} = \frac{28}{5}$

Similarly,

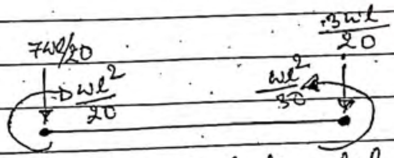
$\frac{\ell}{2}\int_{-1}^{1}(1-\xi)(1-\xi-\xi^2+\xi^3)d\xi = \frac{\ell}{2}\int_{-1}^{1}(1-2\xi+2\xi^3-\xi^4)d\xi = \frac{\ell}{2}(2-0+0-\frac{2}{5}) = \frac{8\ell}{10}$

$\int_{-1}^{1}(1-\xi)(2+3\xi-\xi^3)d\xi = \int_{-1}^{1}(2+\xi-3\xi^2-\xi^3+\xi^4) = 2*2+0-3*\frac{2}{3}-0+\frac{2}{5} = \frac{12}{5}$

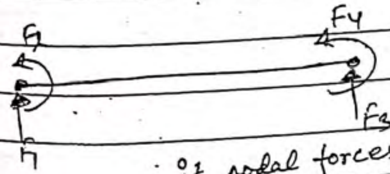
$\int_{-1}^{1}\frac{\ell}{2}(1-\xi)(-1-\xi+\xi^2+\xi^3)d\xi = \frac{\ell}{2}\int_{-1}^{1}(-1+2\xi^2-\xi^4)d\xi = \frac{\ell}{2}\left(-1*2+2*\frac{2}{3}-\frac{2}{5}\right) = \frac{-16\ell}{30}$

Finally,

$$\{F\} = \frac{w\ell}{16}\begin{Bmatrix}\frac{28}{5}\\ \frac{8\ell}{10}\\ \frac{12}{5}\\ -\frac{16\ell}{30}\end{Bmatrix} = \begin{Bmatrix}\frac{7w\ell}{20}\\ \frac{w\ell^2}{20}\\ \frac{3w\ell}{20}\\ \frac{-w\ell^2}{30}\end{Bmatrix}$$



fig(b) :- equivalent nodal forces due to given uniformly varying load of Intensity ω



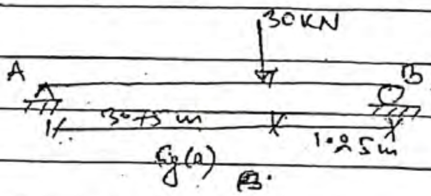
if nodal forces ~~notation~~ direction are as in above fig,
then nodal force vector $\{F\} = \begin{Bmatrix}-7w\ell/20\\ -w\ell^2/20\\ -3w\ell/20\\ w\ell^2/30\end{Bmatrix}$

Q① A simply supported beam AB of span 5 m. is carrying a point load of 30 KN at a distance 3·75 m from left end A. calculate the slopes at A and B and deflection under the lo using one beam element model. Take $EI = 26*10^{12}$ $N mm^2$

$EI = 26*10^{12} N\text{-}mm^2 =$

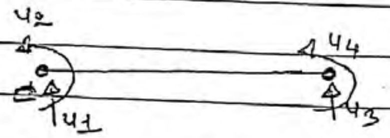
$= 26*10^{12}*10^{-3}*10^{-6}$ $KN m^2$

$= 26*10^{3} KN m^2$

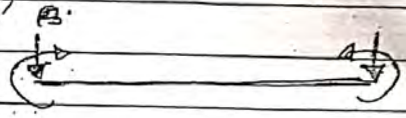


fig(a)

Sol^n



fig(b) nodal displacement notation



fig(c) equivalent nodal forces.

$$\text{stiffness matrix } [K] = \frac{26*10^3}{5^3} \begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ 12 & 30 & -12 & 30 & 1 \\ 30 & 100 & -30 & 50 & 2 \\ -12 & -30 & 12 & -30 & 3 \\ 30 & 50 & -30 & 100 & 4 \end{array}$$

$$\text{~~Elem~~ nodal load vector } \{F^e\} = \begin{Bmatrix} \frac{-Pb}{l} \\ -\frac{Pab^2}{l^2} \\ \frac{-Pa}{l} \\ \frac{Pa^2b}{l^2} \end{Bmatrix} = \begin{Bmatrix} -7.5 \\ -7.0313 \\ -22.5 \\ 21.0938 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$$\text{Global load vector } \{F\} = \begin{Bmatrix} -7.5 + R_A \\ 7.0313 \\ -22.5 + R_B \\ 21.0938 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

~~Element~~ Global equilibrium,

$$[K]\{u\} = \{F\}$$

$$\frac{26*10^3}{5^3}\begin{bmatrix} 12 & 30 & -12 & 30 \\ 30 & 100 & -30 & 50 \\ -12 & -30 & 12 & -30 \\ 30 & 50 & -30 & 100 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \left\{\begin{array}{c|c} -7.5+R_A & 1 \\ -7.0313 & 2 \\ -22.5 & 3 \\ 21.0938 & 4 \end{array}\right.$$

with the columns of $[K]$ numbered 1, 2, 3, 4.

Applying boundary condition, $u_1 = u_3 = 0$, we get

$$\frac{26*10^3}{5^3}\begin{bmatrix} 100 & 50 \\ 50 & 100 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} -7.0313 \\ 21.0938 \end{Bmatrix}$$

on solving,

$$u_2 = -0.0011268 \text{ radian} = \theta_A \quad \underline{\text{Ans}}$$

$$u_4 = 0.0015775 \text{ radian} = \theta_B$$

Deflection under load.

1st Find out ξ when $x = 3.75$ m

$$\xi = \frac{2\cdot(x-x_1)}{(x_2-x_1)} - 1$$

$$\Rightarrow \xi = \frac{2}{5}(3.75-0) - 1 = 0.5$$

We have, $u(\xi) = H_1u_1 + \frac{l}{2}H_2u_2 + H_3u_3 + \frac{l}{2}H_4u_4$

Hence $u(0.5) = H_1*0 + \frac{5}{2}*\frac{1}{4}(1-0.5-0.5^2+0.5^3)(-0.0011268) + H_3*0 + \frac{5}{2}*\frac{1}{4}(-1-0.5+0.5^2+0.5^3)*0.0015775$

$= -1.3736*10^{-3}$ m $= -1.3736$ mm $\underline{\text{Ans}}$

~~QNo2. For beam shown in associated fig:- compute the~~

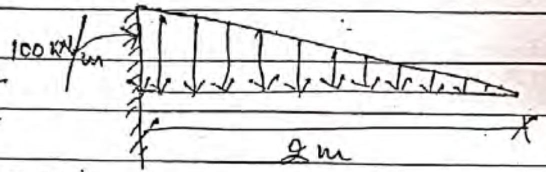
~~d~~ QNo.2. A cantilever of 2m span carries a triangular load of zero intensity at the free end and 100 kN/m at the fixed end. Determine the slope and deflection at the free end. Take $I = 100*10^6 mm^4$ $E = 200 GPa$

$EI = 200*10^9 N/m^2 * 100*10^6 mm^4$

= [illegible]

$= 200*10^9*10^{-3} kN/m^2 * 100*10^6*10^{-12} m^2$

$= 20000 kN\text{-}m^2$



Sol^n We consider a one element formed by two nodes.

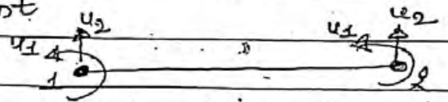


Fig 1- nodal displacement notation

Stiffness matrix

$$[K] = \frac{20000}{2^3} \begin{array}{c|cccc|c} & 1 & 2 & 3 & 4 & \\ \hline & 12 & 12 & -12 & 12 & 1 \\ & 12 & 16 & -12 & 8 & 2 \\ & -12 & -12 & 12 & -12 & 3 \\ & 12 & 8 & -12 & 16 & 4 \end{array}$$

~~Element for~~ nodal load vector

$$\{Fe\} = \begin{Bmatrix} -7wl/20 \\ -wl^2/20 \\ -3wl/20 \\ wl^2/30 \end{Bmatrix} = \begin{Bmatrix} -70 \\ -20 \\ -30 \\ 13.33 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$$\text{Global load vector} = \begin{Bmatrix} -70+R_1 \\ -20+R_2 \\ -30 \\ 13.33 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

Now Global equilibrium,

$[K]\{u\} = \{F\}$

$$\frac{20000}{2^3}\begin{array}{c}\begin{array}{cccc}1 & 2 & 3 & 4\end{array}\\ \begin{bmatrix}12 & 12 & -12 & 12\\ 12 & 16 & -12 & 8\\ -12 & -12 & 12 & -12\\ 12 & 8 & -12 & 16\end{bmatrix}\end{array}\begin{Bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{Bmatrix} = \begin{Bmatrix}-70+R_1\\ -20+R_2\\ -30\\ 13.33\end{Bmatrix}\begin{array}{c}1\\ 2\\ 3\\ 4\end{array}$$

Applying boundary condition, $u_1 = u_2 = 0$, we get

$$\frac{20000}{2^3}\begin{bmatrix}12 & -12\\ -12 & 16\end{bmatrix}\begin{Bmatrix}u_3\\ u_4\end{Bmatrix} = \begin{Bmatrix}-30\\ 13.33\end{Bmatrix}$$

On solving,

$u_3 = -2.6667 \times 10^{-3}\,m = -2.6667\,mm$

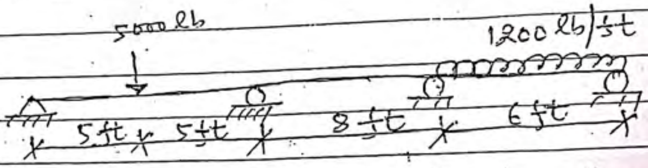
$u_4 = -0.001667$ rad

Hence, slope at the free end $= -0.001667$ radian Ans/

deflection at the free end $= -2.6667$ mm Ans/

QNo.3 For beam shown in associated fig. compute the deflection at the element nodes.

QNo.3 A three span beam is shown in fig. Determine the deflection curve of the beam and evaluate the reaction at the supports.



$$E = 30*10^6 \text{ psi}$$

$$I = 305 \text{ in}^4$$

Sol^n We consider the three elements formed the four nodes.

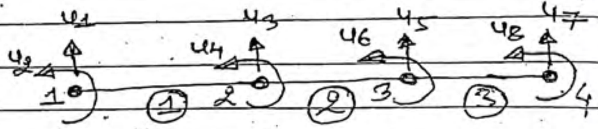


fig:- element, nodal displacement notations

stiffness matrix

For element 1

$$[K^1] = \frac{30*10^6*305}{120^3}\begin{bmatrix} 12 & 720 & -12 & 720 \\ 720 & 57600 & -720 & 28800 \\ -12 & -720 & 12 & -720 \\ 720 & 28800 & -720 & 57600 \end{bmatrix}$$

$$= 9.15 \times 10^9 \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \left[\begin{matrix} 6.95*10^{-6} & 4.167*10^{-4} & -6.95*10^{-6} & 4.167*10 \\ 4.167*10^{-4} & 0.0333 & -4.167*10^{-4} & 0.0167 \\ -6.95*10^{-6} & -4.167*10^{-4} & 6.95*10^{-6} & -4.167*10^{-4} \\ 4.167*10^{-4} & 0.0167 & -4.167*10^{-4} & 0.0333 \end{matrix}\right. \end{array}$$

For element

$$[K^2] = 30*1$$

9

$$= 9.15$$

For element

$$[K^3] =$$

$$= 9.$$

For element 2

$$[k^2] = \frac{30*10^6*305}{96^3}\begin{bmatrix} 12 & 576 & -12 & 576 \\ 576 & 36864 & -576 & 18432 \\ -12 & -576 & 12 & -576 \\ 576 & 18432 & -576 & 36864 \end{bmatrix}$$

$$= 9.15*10^9 \begin{array}{c|c} \begin{matrix} 3 & 4 & 5 & 6 \end{matrix} & \\ \begin{bmatrix} 1.356*10^{-5} & 6.51*10^{-4} & -1.356*10^{-5} & 6.51*10^{-4} \\ 6.51*10^{-4} & 0.04167 & -6.51*10^{-4} & 0.0208 \\ -1.356*10^{-5} & -6.51*10^{-4} & 1.356*10^{-5} & +6.51*10^{-4} \\ 6.51*10^{-4} & 0.0208 & -6.51*10^{-4} & 0.04167 \end{bmatrix} & \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix} \end{array}$$

For element 3

$$[k^3] = \frac{30*10^6*305}{72^3}\begin{bmatrix} 12 & 432 & -12 & 432 \\ 432 & 20736 & -432 & 10368 \\ -12 & -432 & 12 & -432 \\ 432 & 10368 & -432 & 20736 \end{bmatrix}$$

$$= 9.15*10^9 \begin{array}{c|c} \begin{matrix} 5 & 6 & 7 & 8 \end{matrix} & \\ \begin{bmatrix} 3.215*10^{-5} & 1.157*10^{-3} & -3.215*10^{-5} & 1.157*10^{-3} \\ 1.157*10^{-3} & 0.056 & -1.157*10^{-3} & 0.0278 \\ -3.215*10^{-5} & -1.157*10^{-3} & 3.215*10^{-5} & -1.157*10^{-3} \\ 1.157*10^{-3} & 0.0278 & -1.157*10^{-3} & 0.056 \end{bmatrix} & \begin{matrix} 5 \\ 6 \\ 7 \\ 8 \end{matrix} \end{array}$$

On assembling, Global stiffness matrix,

$$[K] = 9.15*10^9 \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \left[\begin{array}{cccccccc} 6.95*10^{-6} & 4.167*10^{-4} & -6.95*10^{-6} & 4.167*10^{-4} & 0 & 0 & 0 & 0 \\ 4.167*10^{-4} & 0.0333 & -4.167*10^{-4} & 0.0167 & 0 & 0 & 0 & 0 \\ -6.95*10^{-6} & -4.167*10^{-4} & 2.051*10^{-5} & 2.343*10^{-4} & -1.356*10^{-5} & 6.51*10^{-4} & 0 & 0 \\ 4.167*10^{-4} & 0.0167 & 2.343*10^{-4} & 0.075 & -6.51*10^{-4} & 0.0208 & 0 & 0 \\ 0 & 0 & -1.356*10^{-5} & -6.51*10^{-4} & 4.571*10^{-5} & 5.06*10^{-4} & -3.215*10^{-5} & 1.157*10^{-3} \\ 0 & 0 & 6.51*10^{-4} & 0.0208 & 5.06*10^{-4} & 0.0978 & -1.157*10^{-3} & 0.0278 \\ 0 & 0 & 0 & 0 & -3.125*10^{-5} & -1.157*10^{-3} & 3.215*10^{-5} & -1.157*10^{-3} \\ 0 & 0 & 0 & 0 & 1.157*10^{-3} & 0.0278 & -1.157*10^{-3} & 0.056 \end{array}\right] \end{array}$$

Nodal load vector:

For element 1

Point load is replaced by following nodal forces

$$\{f^1\} = \begin{Bmatrix} -Pb/l \\ -Pab^2/l^2 \\ -Pa/l \\ Pa^2b/l^2 \end{Bmatrix} = \begin{Bmatrix} -2500 \\ -5000*(5*12)(5*12)^2/(10*12)^2 \\ -2500 \\ 5000*(5*12)^2(5*12)/(10*12)^2 \end{Bmatrix} = \begin{Bmatrix} -2500 \\ -75000 \\ -2500 \\ 75000 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

For element 2, there is no load.

For element 3,
$$\{f^3\} = \begin{Bmatrix} -wl/2 \\ -wl^2/12 \\ -wl/2 \\ wl^2/12 \end{Bmatrix} = \begin{Bmatrix} -3600 \text{ lb} \\ -3600 \text{ lbft} \\ -3600 \text{ lb} \\ 3600 \text{ lbft} \end{Bmatrix} = \begin{Bmatrix} -3600 \\ -43200 \text{ lbin} \\ -3600 \\ 43200 \end{Bmatrix} \begin{matrix} 5 \\ 6 \\ 7 \\ 8 \end{matrix}$$

Now, On Assembling, nodal load vector for whole str
$$\{f\} = \begin{Bmatrix} -2500 \\ -75000 \\ -2500 \\ 75000 \\ -3600 \\ -43200 \\ -3600 \\ 43200 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix}$$

$$\begin{array}{ccc} 7 & 8 & \\ 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \\ -3.245*10^{-5} & 1.157*10^{-3} & 5 \\ -1.157*10^{-3} & 0.0278 & 6 \\ 3.245*10^{-5} & -1.157*10^{-3} & 7 \\ -1.157*10^{-3} & 0.056 & 8 \end{array}$$

es

$$\begin{Bmatrix} -2500 \\ -75000 \\ -2500 \\ 7500 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$$\begin{matrix} 3600 \\ 13200\ lb\text{-}in \\ 600 \\ 3200 \end{matrix} \Bigg\} \begin{matrix} 5 \\ 6 \\ 7 \\ 8 \end{matrix}$$

$$\begin{Bmatrix} -2500 \\ -75000 \\ -2500 \\ 75000 \\ -3600 \\ -43200 \\ -3600 \\ 43200 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix}$$

let, R_1, R_3, R_5 & R_7 be the reaction at ~~suppo~~ corresponding supports, then

Global load vector,
$$\{F\} = \begin{Bmatrix} -2500+R_1 \\ -75000 \\ -2500+R_3 \\ 75000 \\ -3600+R_5 \\ -43200 \\ -3600+R_7 \\ 43200 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix}$$

and nodal displacement
$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix}$$

Applying boundary condition, $u_1 = u_3 = u_5 = u_7$, we get following form,

$$9.15*10^9 \begin{bmatrix} 0.0333 & 0.0167 & 0 & 0 \\ 0.0167 & 0.075 & 0.0208 & 0 \\ 0 & 0.0208 & 0.0978 & 0.0278 \\ 0 & 0 & 0.0278 & 0.056 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_4 \\ u_6 \\ u_8 \end{Bmatrix} = \begin{Bmatrix} -75000 \\ 75000 \\ -43200 \\ 43200 \end{Bmatrix}$$

$0.0333\psi_2 + 0.0167\psi_4 = -75000/9.15*10^9$ —— ①

$0.0167\psi_2 + 0.075\psi_4 + 0.0208\psi_6 = 75000/9.15*10^9$ —— ②

$0.0208\psi_4 + 0.0978\psi_6 + 0.0278\psi_8 = -43200/9.15*10^9$ — ③

$0.0278\psi_6 + 0.056\psi_8 = 43200/9.15*10^9$ — ④

From ① $0.0334\psi_2 = \dfrac{-75000}{9.15*10^9} - 0.0167\psi_4$

$0.0333\psi_2 = -8.197*10^{-6} - 0.0167\psi_4$

$\psi_2 = -2.46*10^{-4} - 0.5015\psi_4$ — ⑤

From ④ $0.056\psi_8 = \dfrac{43200}{9.15*10^9} - 0.0278\psi_6$

$\psi_8 = 8.43*10^{-5} - 0.496\psi_6$ — ⑥

Substituting the value of ⑤ in ② and ⑥ in ③, we get,

$0.0167\left(-2.46*10^{-4} - 0.5015\psi_4\right) + 0.075\psi_4 + 0.0208\psi_6 = \dfrac{75000}{9.15*10^9}$

$-8.375*10^{-3}\psi_4 + 0.075\psi_4 + \cancel{0.208}\,0.0208\psi_6$

$= 8.197*10^{-6} + 4.108*10^{-6}$

$0.0666\psi_4 + 0.0208\psi_6 = 1.2305*10^{-5}$ —— ⑦

$0.0208\psi_4 + 0.0978\psi_6 + 0.0278\left(8.43*10^{-5} - 0.496\psi_6\right) = \dfrac{-43200}{9.15*10^9}$

or, $0.0208\psi_4 + 0.0978\psi_6 - 0.0138\psi_6 = -4.72*10^{-6} - 2.344*10^{-6}$

$0.0208\psi_4 + 0.084\psi_6 = -7.064*10^{-6}$ —— ⑧

solving ⑦ & ⑧,

$\psi_4 = 2.287*10^{-4}$

$\psi_6 = -1.407*10^{-4}$ radian

From ⑤ $\psi_2 = -2.46*10^{-4} - 0.5015*2.287*10^{-4} = -3.61*10^{-4}$ radian

From ⑥ $\psi_8 = 8.43*10^{-5} - 0.496*\left(-1.407*10^{-4}\right) = 1.541*10^{-4}$

For element 1

deflection under load, load is situated at $\xi = 0$,

We have, $v(\xi) = H_1\psi_1 + \frac{l}{2}H_2\psi_2 + H_3\psi_3 + \frac{l}{2}H_4\psi_4$

$$v(0) = 0 + \left(\frac{120}{2}\right)\left(\frac{1}{4}\right)\left(-3.61*10^{-4}\right) + 0 + \frac{120}{2}\left(-\frac{1}{4}\right)\left(2.287*10^{-4}\right)$$

$$= -0.00885 \text{ in}$$

For element 3.

$\psi_1 = \psi_5 = 0$, $\psi_2 = \psi_6 = -1.407*10^{-4}$, $\psi_3 = \psi_7 = 0$, $\psi_4 = \psi_8 = 1.541*10^{-4}$

deflection at mid point

$$v(0) = 0 + \frac{72}{2}\left(\frac{1}{4}\right)\left(-1.407*10^{-4}\right) + 0 + \frac{72}{2}\left(\frac{-1}{4}\right)\left(1.541*10^{-4}\right)$$

$$= \underline{-0.00265 \text{ in}}$$

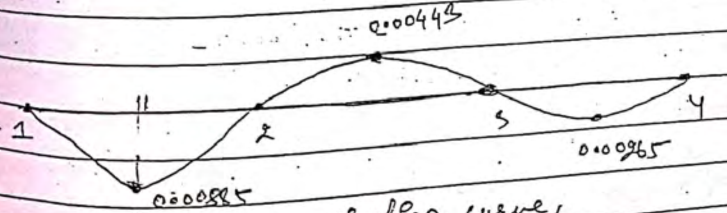
for element 2.

deflection at mid point

$\psi_1 = \psi_3 = 0$, $\psi_2 = \psi_4 = 2.287*10^{-4}$ $\psi_3 = \psi_5 = 0$, $\psi_4 = \psi_6 = -1.407*10^{-4}$

$$v(0) = 0 + \frac{96}{2}\left(\frac{1}{4}\right)\left(2.287*10^{-4}\right) + \frac{96}{2}\left(\frac{-1}{4}\right)\left(-1.407*10^{-4}\right)$$

$$= 0.00443 \text{ in}$$



Deflection curve,

Find out Support reaction

plane Frame element

→ Similar to the beams except that axial load and axial deformation are present. A superposition of bar and beam gives a finite element that is known as frame element. Members of frame structure are connected by rigid connection (eg. welded or riveted) and therefore axial, transverse forces and bending moments are developed in the members.

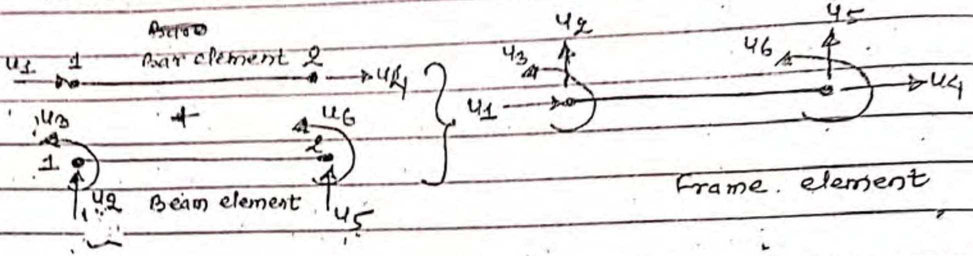
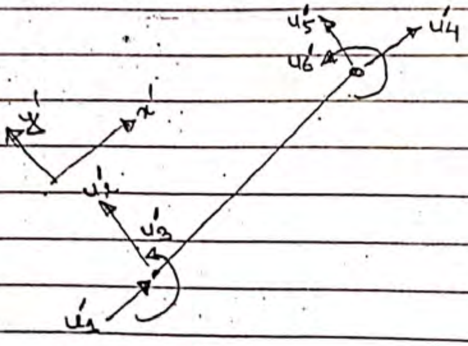
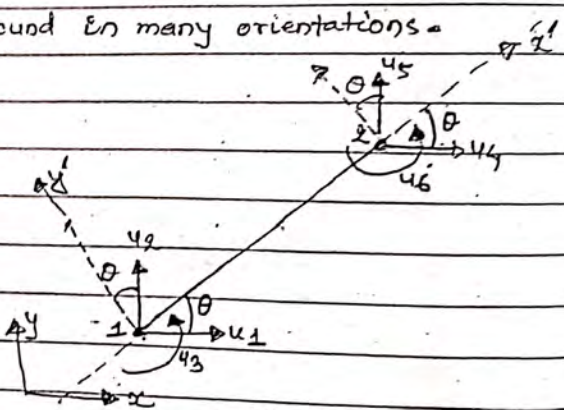


Fig:- Superposition of bar and beam element to obtain a frame element

Frame elements are found in many orientations.





Fig(a) plane frame element and nodal displacement ~~[illegible]~~ notations in local coordinate system

Fig(b) plane frame ① element and corresponding nodal displacement ~~notations~~ in global coordinate system

The nodal displacement vector in global coordinate system,

$$\{u\} = \{u_1, u_2, u_3, u_4, u_5, u_6\}^T$$

The nodal displacement in local coordinate system.

$$\{u'\} = \{u'_1, u'_2, u'_3, u'_4, u'_5, u'_6\}$$

Recognizing that $u'_3 = u_3$ and $u'_6 = u_6$

From fig ⓐ & ⓑ,

$$u'_1 = u_1 \cos\theta + u_2 \sin\theta$$

$$u'_2 = -u_1 \sin\theta + u_2 \cos\theta$$

$$u'_3 = u_3$$

in vector form

$$\begin{Bmatrix} u'_1 \\ u'_2 \\ | \\ | \\ | \\ u'_6 \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 & 0 & 0 \\ -m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & 0 \\ 0 & 0 & 0 & -m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ | \\ | \\ | \\ u_6 \end{Bmatrix}$$

Where $l = \cos\theta$
$m = \sin\theta$
as in truss element.

$$\{u'\} = [L]\{u\}$$

$[L] =$ transformation matrix

It is observed that u'_2, u'_3 and u'_5, u'_6 are like the beam degress of freedom while u'_1 and u'_4 are similar to the ~~so~~ bar element. Combining two stiffness and arranging in proper location, we get $[K]$ for frame element

~~$$[K] = \begin{array}{cccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & \\ \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 & 1 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & \frac{12EI}{L^3} & & 2 \\ & & & & & & 3 \\ & & & & & & 4 \\ & & & & & & 5 \\ & & & & & & 6 \end{array}$$~~

We have, for bar element,

$$[K]_{bar} = \begin{array}{cc|c} 1 & 4 & \\ \frac{EA}{L} & -\frac{EA}{L} & 1 \\ -\frac{EA}{L} & \frac{EA}{L} & 4 \end{array}$$

For beam element

$$[K]_{beam} = \begin{array}{cccc|c} 2 & 3 & 5 & 6 & \\ \frac{12EI}{\ell^3} & \frac{6EI}{\ell^2} & \frac{-12EI}{\ell^3} & \frac{6EI}{\ell^2} & 2 \\ 6EI/\ell^2 & \frac{4EI}{\ell^2} & -\frac{6EI}{\ell^2} & \frac{2EI}{\ell} & 3 \\ \frac{-12EI}{\ell^3} & \frac{-6EI}{\ell^2} & \frac{12EI}{\ell^3} & \frac{-6EI}{\ell^2} & 5 \\ \frac{6EI}{\ell^2} & \frac{2EI}{\ell} & \frac{-6EI}{\ell^2} & \frac{4EI}{\ell} & 6 \end{array}$$

On Assembling $[K]_{bar}$ and $[K]_{beam}$, we get the stiffness matrix for frame element in local coordinate system.

$$[K']_{frame} = \begin{array}{cccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & \\ \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 & 1 \\ 0 & \frac{12EI}{\ell^3} & \frac{6EI}{\ell^2} & 0 & \frac{-12EI}{\ell^3} & \frac{6EI}{\ell^2} & 2 \\ 0 & \frac{6EI}{\ell^2} & \frac{4EI}{\ell^2} & 0 & \frac{-6EI}{\ell^2} & \frac{2EI}{\ell} & 3 \\ -\frac{EA}{\ell} & 0 & 0 & \frac{EA}{\ell} & 0 & 0 & 4 \\ 0 & \frac{-12EI}{\ell^3} & \frac{-6EI}{\ell^2} & 0 & \frac{12EI}{\ell^3} & \frac{-6EI}{\ell^2} & 5 \\ 0 & \frac{6EI}{\ell^2} & \frac{2EI}{\ell} & 0 & \frac{-6EI}{\ell^2} & \frac{4EI}{\ell} & 6 \end{array}$$

Stiffness matrix in global coordinate system

$$\boxed{[K] = [L]^T [K'] [L]}$$

as in truss element

https://civinnovate.com/civil-engineering-notes/

No.1 The frame of figure is composed of identical having a 1-in square cross section and modulus of ticity of 10*10^6 psi. The supports at O and C are to be idered completely fixed. The horizontal beam is subjected a uniform load of intensity 10 lb/in as shown. Use o beam-axial elements to compute the displacements d rotation at B.

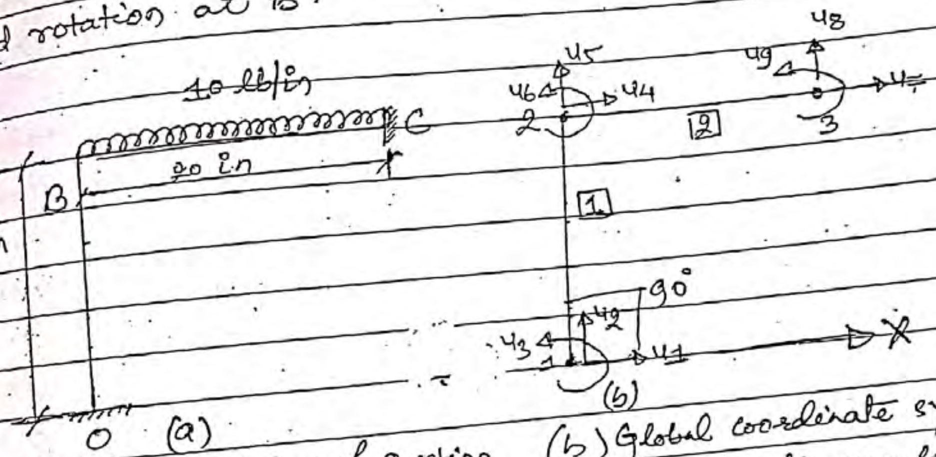


Fig (1) Frame of Question (b) Global coordinate system and displacement numbering

Sol) c/s Area = $1*1 = 1 in^2$

Area moment of intertia about x-axis

$$\boxed{I = \frac{bh^3}{12}} = \frac{1*1^3}{12} = 0.083 in^4$$

The axial stiffness = $\frac{EA}{l} = \frac{10*10^6*1}{20} = 5*10^5$ lb/in

bending stiffness = $\frac{EI}{l^3} = \frac{10*10^6*0.083}{20^3} = 104.2$ lb/in

Denoting member OB as element 1 and member BC as element 2, the stiffness matrices in the local coordinate system are identical and given by new

$$[k^{(1)}] = [k^{(2)}] = [k'^{(2)}] = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}\left[\begin{array}{cccccc|c} 4 & 5 & 6 & 7 & 8 & 9 & \\ 5*10^5 & 0 & 0 & -5*10^5 & 0 & 0 & 4 \\ 0 & 1250.4 & 12504 & 0 & -1250.4 & 12504 & 5 \\ 0 & 12504 & 166720 & 0 & -12504 & 83360 & 6 \\ -5*10^5 & 0 & 0 & 5*10^5 & 0 & 0 & 7 \\ 0 & -1250.4 & -12504 & 0 & 12504 & -12,504 & 8 \\ 0 & 12,504 & 83360 & 0 & -12504 & 166720 & 9 \end{array}\right]$$

Choosing the global coordinate system and displacement numbering as in fig (b). We observe that element 2 requires no transformation as its local coordinate system is aligned with global system. However element 1 requires transformation

$$[k^{(2)}] = [k'^{(2)}]$$

We have, element ① is aligned at an angle 90° to the horizontal

We have, $[k^{(1)}] = [L]^T [k'^{(1)}] [L]$ — ①

$$[L] = \begin{bmatrix} l & m & 0 & 0 & 0 & 0 \\ -m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & 0 \\ 0 & 0 & 0 & -m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

From ①

$$[k^{(1)}] = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5*10^5 & 0 & 0 & -5*10^5 & 0 & 0 \\ 0 & 1250.4 & 12504 & 0 & -1250.4 & 12504 \\ 0 & 12504 & 166720 & 0 & -12504 & 83360 \\ -5*10^5 & 0 & 0 & 5*10^5 & 0 & 0 \\ 0 & -1250.4 & -12504 & 0 & 1250.4 & -12,504 \\ 0 & 12504 & 83360 & 0 & -12504 & 166720 \end{bmatrix}$$

$$[k^{(1)}] = \begin{bmatrix} 0 & -1250.4 & -12504 & 0 & 1250.4 & -12504 \\ 5*10^5 & 0 & 0 & -5*10^5 & 0 & 0 \\ 0 & 12504 & 166720 & 0 & -12504 & 83360 \\ 0 & 1250.4 & 12504 & 0 & -1250.4 & 12504 \\ -5*10^5 & 0 & 0 & 5*10^5 & 0 & 0 \\ 0 & 12504 & 83360 & 0 & -1250.4 & 166720 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[k^{(1)}] = \begin{array}{cccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline 1250.4 & 0 & -12504 & -1250.4 & 0 & -12504 & 1 \\ 0 & 5*10^5 & 0 & 0 & -5*10^5 & 0 & 2 \\ -12504 & 0 & -166720 & 12504 & 0 & 83,360 & 3 \\ -1250.4 & 0 & 12504 & 1250.4 & 0 & 12,504 & 4 \\ 0 & -5*10^5 & 0 & 0 & 5*10^5 & 0 & 5 \\ -12504 & 0 & 83,360 & 12504 & 0 & 166720 & 6 \end{array}$$

On Assembling, we get Global stiffness matrix

$$[K] = \begin{array}{ccccccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \\ \hline 1250.4 & 0 & -12,504 & -1250.4 & 0 & -12504 & 0 & 0 & 0 & 1 \\ 0 & 5*10^5 & 0 & 0 & -5*10^5 & 0 & 0 & 0 & 0 & 2 \\ -12504 & 0 & 166720 & 12504 & 0 & 833360 & 0 & 0 & 0 & 3 \\ -1250.4 & 0 & 12504 & 501250.4 & 0 & 12504 & -5*10^5 & 0 & 0 & 4 \\ 0 & -5*10^5 & 0 & 0 & 501250.4 & 12504 & 0 & -1250.4 & 12504 & 5 \\ -12504 & 0 & 83,360 & 12504 & 12504 & 333440 & 0 & -12504 & 83360 & 6 \\ 0 & 0 & 0 & -5*10^5 & 0 & 0 & 5*10^5 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & -1250.4 & -12504 & 0 & 1250.4 & -12,504 & 8 \\ 0 & 0 & 0 & 0 & 12,504 & \overset{\cancel{125}}{83,360} & 0 & -12,504 & 166,720 & 9 \end{array}$$

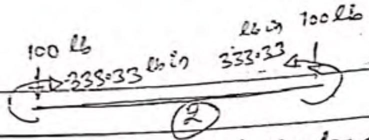


Fig equivalent load for element 2

Nodal load vector,

$\{f^1\} = \{0\}$, ∵ there is no load.

$$\{f^2\} = \begin{array}{c|c} \begin{Bmatrix} -100 \\ -333.33 \\ -100 \\ 333.33 \end{Bmatrix} & \begin{matrix} 5 \\ 6 \\ 8 \\ 9 \end{matrix} \end{array}$$

Let, R_{1x}, R_{1y} & M_1 be the reactions at support O
and R_{3x}, R_{3y} & M_3 be the " " " C.

then Global load vector

$$\{F\} = \begin{array}{cc} \begin{Bmatrix} R_{1x} \\ R_{1y} \\ M_1 \\ 0 \\ -100 \\ -333.33 \\ R_{3x} \\ -100+R_{3y} \\ 333.33+M_3 \end{Bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{matrix} \end{array}$$

Global equilibrium,

$$[K]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} = \begin{array}{cc} \begin{Bmatrix} R_{1x} \\ R_{1y} \\ M_1 \\ 0 \\ -100 \\ -333.33 \\ R_{3x} \\ -100+R_{3y} \\ 333.33+M_3 \end{Bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{matrix} \end{array} \longrightarrow ②$$

Applying boundary conditions $u_1 = u_2 = u_3 = u_7 = u_8 = u_9 = 0$ in this eqn, we get the following reduced form.

501250·4	0
0	50
12504	1.

On

Reaction com

computed di

Above com

Now, local

$\{u'\} =$

u'_1
u'_2
u'_3
u'_4
u'_5
u'_6

$$\begin{bmatrix} 501250.4 & 0 & 12504 \\ 0 & 501250.4 & 12504 \\ 12504 & 12504 & 333440 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -100 \\ -333.33 \end{Bmatrix}$$

on solving

$u_4 = 2.47974 * 10^{-5}$ in

$u_5 = -1.074704 * 10^{-4}$ in Ans

$u_6 = -$ ~~0.01008 radian~~ $-9.94058 * 10^{-4}$ rad

Reaction compotents can be computed by substituting the computed displacements in global equilibrium equation eq ②.

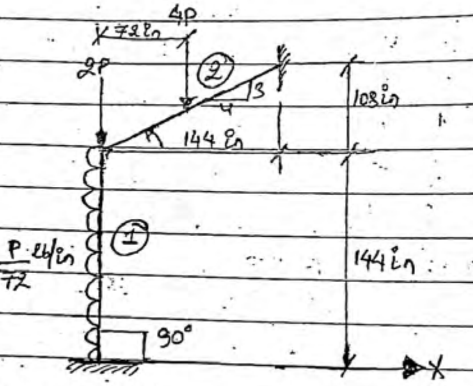
Above computed displacements are global displacements.

Now, local displacements

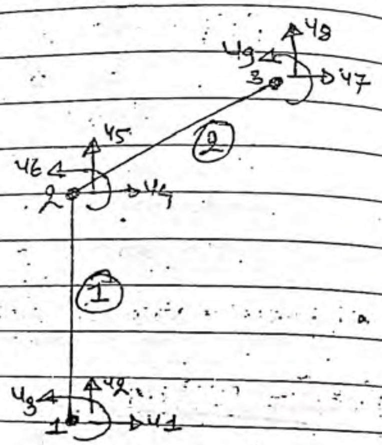
$\boxed{\{u'\} = [L]\{u\}}$ only for element 1

$$\begin{Bmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \\ u_5' \\ u_6' \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1.74704 * 10^{-4} \\ -2.4794 * 10^{-5} \\ -9.94058 * 10^{-4} \end{Bmatrix}$$

Q.No.2. The frame structure shown in fig is to be analysed for displa and forces. Both members of the structure are made up of Same material (E) and have the same geometric properties(A, I
The element stiff Take, $E = 10^6$ psi $A = 10\ in^2$ $I = 10\ in^4$.



Fig(a) Geometry and leading



Fig(b) displacements in plane frame structure in global coordinate system.

For element 1

$l = 144$ in $\cos\theta = 0$ $\sin\theta = 1$

$\frac{EA}{l} = \frac{10^6 * 10}{144} = 10^5 * 0.6944$, $\frac{6EI}{l^2} = \frac{6*10^6*10}{144^2} = 0.0289$

$\frac{12EI}{l^3} = \frac{12*10^6*10}{144^3} = 10^5 * 0.0004$, $\frac{4EI}{l} = \frac{4*10^6*10}{144} = 2.7778$

$2EI/l = \frac{2*10^6*10}{144} = 1.3889$

$$[K'^{(1)}] = 10^5 \begin{bmatrix} 0.6944 & 0 & 0 & -0.6944 & 0 & 0 \\ 0 & 0.0004 & 0.0289 & 0 & -0.0004 & 0.0289 \\ 0 & 0.0289 & 2.7778 & 0 & -0.0289 & 1.3889 \\ -0.6944 & 0 & 0 & 0.6944 & 0 & 0 \\ 0 & -0.0004 & -0.0289 & 0 & 0.0004 & -0.0289 \\ 0 & 0.0289 & 1.3889 & 0 & -0.0289 & 2.779 \end{bmatrix}$$

Element 1 requires
at an angle of 90°

Now, transformation mat

$$\therefore [L] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 \end{bmatrix}$$

Stiffness matrix in g

$$[K^{(1)}] = [L]^T [K'][L]$$

$$\text{or, } [K^{(2)}] = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{or, } [K^{(1)}] = 10^5 \begin{bmatrix} 0 & -0.0004 \\ 0.6944 & 0 & 0 \\ 0 & 0.0289 \\ 0 & 0.0004 \\ -0.6944 & 0 \\ 0 & 0.0289 \end{bmatrix}$$

Element 1 requires transformation because it is oriented at an angle of 90° to the global coordinate.

transformation matrix

$$[L] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

tiffness matrix in global coordinate system,

$$^{(1)}] = [L]^T [K'] [L]$$

$$] = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} 10^{5} \begin{bmatrix} 0.6944 & 0 & 0 & -0.6944 & 0 & 0 \\ 0 & 0.0004 & 0.0289 & 0 & -0.0004 & 0.0289 \\ 0 & 0.0289 & 2.7778 & 0 & -0.0289 & 1.3889 \\ -0.6944 & 0 & 0 & 0.6944 & 0 & 0 \\ 0 & -0.0004 & -0.0289 & 0 & 0.0004 & -0.0289 \\ 0 & 0.0289 & 1.3889 & 0 & -0.0289 & 2.7778 \end{bmatrix} [L]$$

$$= 10^{5} \begin{bmatrix} 0 & -0.0004 & -0.0289 & 0 & 0.0004 & -0.0289 \\ 0.6944 & 0 & 0 & -0.6944 & 0 & 0 \\ 0 & 0.0289 & 2.7778 & 0 & -0.0289 & 1.3889 \\ 0 & 0.0004 & 0.0289 & 0 & -0.0004 & -0.0289 \\ -0.6944 & 0 & 0 & 0.6944 & 0 & 0 \\ 0 & 0.0289 & 1.3889 & 0 & -0.0289 & 2.7778 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\frac{[K^1]}{10^5} = \begin{array}{cccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline 0.0004 & 0 & -0.0289 & -0.0004 & 0 & -0.0289 & 1 \\ 0 & 0.6944 & 0 & 0 & -0.6944 & 0 & 2 \\ -0.0289 & 0 & 2.778 & 0.0289 & 0 & 1.3889 & 3 \\ -0.0004 & 0 & 0.0289 & 0.0004 & 0 & 0.0289 & 4 \\ 0 & -0.6944 & 0 & 0 & 0.6944 & 0 & 5 \\ -0.0289 & 0 & 1.3889 & 0.0289 & 0 & 2.7778 & 6 \end{array}$$

For element 2.

$$l = \sqrt{108^2 + 144^2} = 180 \text{ in}$$

$$l = \cos\theta = \frac{x_2 - x_1}{l} = \frac{144-0}{180} = 0.8 \qquad \sin\theta = \frac{y_2 - y_1}{l} = \frac{108-0}{180} = 0.6 = m$$

$$\frac{EA}{l} = \frac{10^6 * 10}{180} = 10^5 * 0.5556 \qquad \frac{12EI}{l^3} = \frac{12 * 10^6 * 10}{180^3} = 10^5 * 0.0002$$

$$\frac{6EI}{l^2} = \frac{6 * 10^6 * 10}{180^2} = 10^5 * 0.0185 \qquad \frac{4EI}{l} = \frac{4 * 10^6 * 10}{180} = 2.2222 * 10^5$$

$$\frac{2EI}{l} = \frac{2 * 10^6 * 10}{180} = 1.1111 * 10^5$$

stiffness matrix in local coordinate system

$$[K'^{(2)}] = 10^5 \begin{bmatrix} 0.5556 & 0 & 0 & -0.5556 & 0 & 0 \\ 0 & 0.0002 & 0.0185 & 0 & -0.0002 & 0.0185 \\ 0 & 0.0185 & 2.2222 & 0 & -0.0185 & 1.1111 \\ -0.5556 & 0 & 0 & 0.5556 & 0 & -0 \\ -0 & -0.0002 & -0.0185 & 0 & -0.0002 & -0.0185 \\ 0 & 0.0185 & 1.1111 & 0 & -0.0185 & 2.2222 \end{bmatrix}$$

This element requires transformation
transformation matrix

$$[L] = \begin{bmatrix} 0.8 & 0.6 & 0 & 0 & 0 & 0 \\ -0.6 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0.6 & 0 \\ 0 & 0 & 0 & -0.6 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now stiffness matrix in global coordinate system

$$[K] = [L]^T [k] [L]$$

$$= \begin{bmatrix} 0.8 & -0.6 & 0 & 0 & 0 & 0 \\ 0.6 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & -0.6 & 0 \\ 0 & 0 & 0 & 0.6 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} 10^5 \begin{bmatrix} 0.5556 & 0 & 0 & -0.5556 & 0 & 0 \\ 0 & 0.0002 & 0.0185 & 0 & -0.0002 & 0.0185 \\ 0 & 0.0185 & 2.2222 & 0 & -0.0185 & 1.111 \\ -0.5556 & 0 & 0 & 0.5556 & 0 & 0 \\ 0 & -0.0002 & -0.0185 & 0 & 0.0002 & -0.0185 \\ 0 & 0.0185 & 1.111 & 0 & -0.0185 & 2.222 \end{bmatrix} [L]$$

$$= 10^5 \begin{bmatrix} 0.4445 & -0.0001 & -0.0111 & -0.4445 & 0.0001 & -0.0111 \\ 0.3334 & 0.0002 & 0.0148 & -0.3334 & -0.0002 & 0.0148 \\ 0 & 0.0185 & 2.2222 & 0 & -0.0185 & 1.111 \\ -0.4445 & 0.0001 & 0.0111 & 0.4445 & -0.0001 & 0.0111 \\ -0.3334 & -0.0002 & -0.0148 & 0.3334 & 0.0002 & -0.0148 \\ 0 & 0.0185 & 1.111 & 0 & -0.0185 & 2.2222 \end{bmatrix} \begin{bmatrix} 0.8 & 0.6 & 0 & 0 & 0 & 0 \\ -0.6 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0.6 & 0 \\ 0 & 0 & 0 & -0.6 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$-er\left[k^{(2)}\right] = 10^5 \begin{array}{cccccc|c} 4 & 5 & 6 & 7 & 8 & 9 & \\ 0.3556 & -0.2666 & -0.0111 & -0.3556 & -0.2666 & -0.0111 & 4 \\ 0.2666 & 0.2001 & 0.0148 & -0.2666 & -0.2001 & 0.0148 & 5 \\ -0.0111 & 0.0148 & 2.2222 & 0.0111 & -0.0148 & 1.1111 & 6 \\ -0.3556 & -0.2666 & 0.0111 & 0.3556 & 0.2666 & 0.0111 & 7 \\ -0.2666 & -0.2001 & -0.0148 & -0.2666 & -0.2001 & -0.0148 & 8 \\ -0.0111 & 0.0148 & 1.1111 & 0.0111 & -0.0148 & 2.2222 & 9 \end{array}$$

On Assembling,

Global stiffness matrix

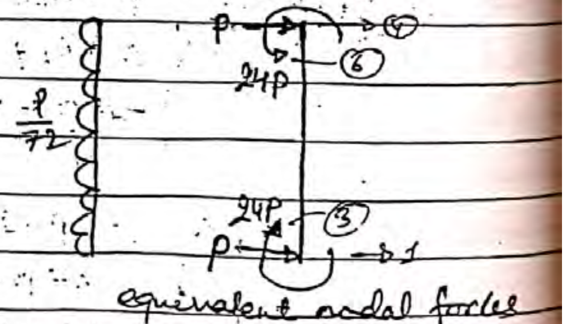
$$[K] = 105 \begin{array}{ccccccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \\ 0.0064 & 0 & -0.0289 & -0.004 & 0 & -0.0289 & 0 & 0 & 0 & 1 \\ 0 & 0.6944 & 0 & 0 & -0.6944 & 0 & 0 & 0 & 0 & 2 \\ -0.0289 & 0 & 2.7778 & 0.0289 & 0 & 1.3889 & 0 & 0 & 0 & 3 \\ -0.004 & 0 & 0.0289 & 0.356 & 0.2666 & 0.0178 & -0.3556 & -0.2666 & -0.0111 & 4 \\ 0 & -0.6944 & 0 & 0.2666 & 0.8945 & 0.0148 & -0.2666 & -0.2001 & 0.0148 & 5 \\ -0.0289 & 0 & 1.389 & 0.0178 & 0.0148 & 5 & 0.0111 & -0.0148 & 1.1111 & 6 \\ 0 & 0 & 0 & -0.3556 & -0.2666 & 0.0111 & 0.3556 & 0.2666 & 0.0111 & 7 \\ 0 & 0 & 0 & -0.2666 & -0.2001 & -0.0148 & 0.2666 & 0.2001 & -0.0148 & 8 \\ 0 & 0 & 0 & -0.0111 & -0.0148 & 1.1111 & 0.0111 & -0.0148 & 2.2222 & 9 \end{array}$$

Applying boundary condition $u_1 = u_2 = u_3 = u_7 = u_8 = u_9 = 0$

Now, Nodal load vector

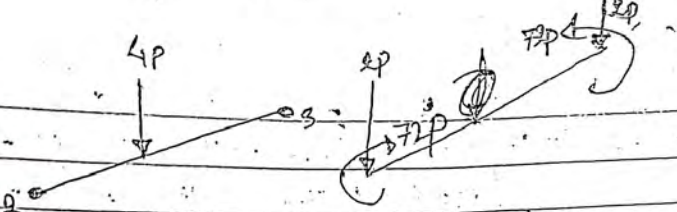
For element 1

$$\{f^1\} = \begin{Bmatrix} wl/2 \\ -wl^2/12 \\ wl/2 \\ wl^2/12 \end{Bmatrix} = \begin{Bmatrix} P \\ -24P \\ P \\ 24P \end{Bmatrix} \begin{matrix} 1 \\ 3 \\ 4 \\ 6 \end{matrix}$$



equivalent nodal forces

For element 2.



equivalent nodal forces

$$\{f_g\} = \begin{Bmatrix} -Pb/l \\ -Pab^2/l^2 \\ -pa/l \\ pa^2b/l \end{Bmatrix} = \begin{Bmatrix} -2p \\ -72p \\ -2p \\ 72p \end{Bmatrix} \begin{matrix} 5 \\ 6 \\ 8 \\ 9 \end{matrix}$$

Load vector due to externally applied load at node

$$\{f_{ex}\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2p \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{matrix}$$

let R_{1x}, R_{1y} and M_1 be the support ~~moment~~ reaction at 1
R_{3x}, R_{3Y} and M_3 be the " reaction at 2,
then, ~~Global load vect~~ then, On Assembling, global load vector

$$\{F\} = \begin{Bmatrix} P+R_{1x} \\ 0+R_{1y} \\ -24P+M_1 \\ P \\ -2p-2P \\ 24p-72p \\ 0+R_{3x} \\ -2p+R_{3Y} \\ 72p+M_3 \end{Bmatrix} = \begin{Bmatrix} P+R_{1x} \\ R_{1y} \\ -24p+M_1 \\ P \\ -4P \\ -48P \\ R_{3x} \\ -2p+R_{3Y} \\ 72p+M_3 \end{Bmatrix}$$

Now, Global equilibrium

$$[K]\{u\} = \{F\}$$

$$[K]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} = \begin{array}{l} \begin{Bmatrix} P+R_{1X} \\ R_{1Y} \\ -24P+M_1 \\ P \\ -4P \\ -48P \\ R_{3X} \\ -2P+R_{3Y} \\ 72P+M_3 \end{Bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$$

Applying boundary condition, $u_1=u_2=u_3=u_6=u_7=u_8=u_9=0$, we get

$$10^5\begin{bmatrix} 0.356 & 0.2666 & 0.0178 \\ 0.2666 & 0.8945 & 0.0148 \\ 0.0178 & 0.0148 & 5 \end{bmatrix}\begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} P \\ -4P \\ -48P \end{Bmatrix}$$

Let $p = 1$ lb.

On solving,

$u_4 = 0.8390 * 10^{-4}$ in

$u_5 = -0.6812 * 10^{-4}$ in

$u_6 = -0.9610 * 10^{-4}$ rad

Ans

Compute the reaction from global equilibrium eqn.

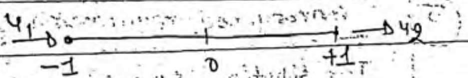
physical problem.

necessity of interpolation function depends upon the solution accuracy and convergence of finite element analysis to the exact solution of a general field problem.

Lagrange Interpolation function

When the interpolation functions are derived to interpolate the function values only and not the derivatives of the function, they are known as the Lagrange interpolation function.

For example for bar element



Shape functions,

$N_1 = \frac{1-\xi}{2}$ and $N_2 = \frac{1+\xi}{2}$

the linear displacement field within element field,

$u(\xi) = N_1 u_1 + N_2 u_2$

Here, shape function or interpolation functions are ~~within~~ only used to ~~interpla~~ interpolate displacement within element and not their derivatives.

Hermite Interpolation function

When the function and its derivatives are interpolated, the resulting interpolation functions are known as Hermite interpolation function.

For example For beam element

$$u(x) = u_1 H_1(x) + H_2\left(\frac{dy}{dx}\right)_1 + u_3 H_3 + H_4\left(\frac{du}{dx}\right)_2$$

~~Here Interpolation function,~~ $H_1, H_2, H_3 + H_4$ are used to interpolate displacement u as well as its derivatives $\frac{du}{dx}$ at the nodes.

① **Compatibility Requirement**

This means that displacements (variable) within the elements and across the element boundaries must be continuous, differentiable. Physically compatibility assures that no gaps between the elements.

② **Completeness requirements**

→ The completeness requirement means that displacement field within structural element should be complete polynomial i.e. include all lower-order terms up to highest order used.

③ **convergence requirement**

The solution should converge to exact solution of general field problem.

4.3.1 PASCAL TRIANGLE

Higher order triangular elements (i.e. triangular elements with interpolation functions of higher degree) can be systematically developed with the help of the so-called Pascal's triangle contains the term of polynomials of various degree in two coordinates x and y as shown in fig.

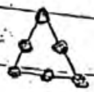
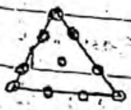
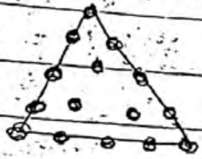
		Polynomial	
① 1	0	1	
x y	1	3	
x^2 xy y^2	2	6	
x^3 x^2y xy^2 y^3	3	10	
x^4 x^3y x^2y^2 xy^3 y^4	4	15	
x^5 x^4y x^3y^2 x^2y^3 xy^4 y^5	5	21	 Fig not shown

Fig:- Top six row's Pascal's triangle

For eg. triangular element of order 2 (i.e. degree of polynomial is 2) contains six nodes can be seen the 3rd row of Pascal's triangle. The polynomial involves six constants.

The polynomial is given by (quadratic)

$$u(x,y) = a_1 + a_2 x + a_3 y + a_4 xy + a_5 x^2 + a_6 y^2$$

Polynomial Function

Polynomial is the most common form of displacement model. This form is widely used because

1. It is easy to handle mathematics of polynomial.
2. A polynomial of arbitrary order permits a recognisable approximation to the solution

p.t.o

$+ a_2 x + a_3 x + ---- a_{n+1} x^n$ is a polynomial of order n and $a_1 ---- a_{n+1}$ are called generalized coordinates.

→ The generalised coordinates $a_1 ---- a_{n+1}$ do not represent displacements rather they represent linear combinations of displacements and slopes as well.

$$u(x) = [N]\{\underline{a}\}$$

General polynomial in two dimensions,

$$u(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 --- a_m y^n$$

$$v(x,y) = a_{m+1} + a_{m+2} x + a_{m+3} y + --- + a_{2m} y^n$$

→ Each of the above general polynomials forms of displacement models can be truncated to any desired degree to give constant, linear, quadratic cubic and higher order patterns.

For one dimensional

$u(x) = a_1$ (constant), $u(x) = a_1 + a_2 x$ (linear)

$u(x) = a_1 + a_2 x + a_3 x^2$ (quadratic), $u(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$ (cubic

For two dimension

$u(x) = a_1$, $v(x) = a_1$ (constant); $u(x) = a_1 + a_2 x + a_3 y$, $v(x) = a_4 + a_5 x + a_6 y$ (linear)

$$u(x) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 x^2$$
$$v(x) = a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} xy + a_{12} y^2$$
(quadratic model)

The usual procedure is to choose the same number of generalized coordinates as that of no. of degrees of freedom.

1 — constant

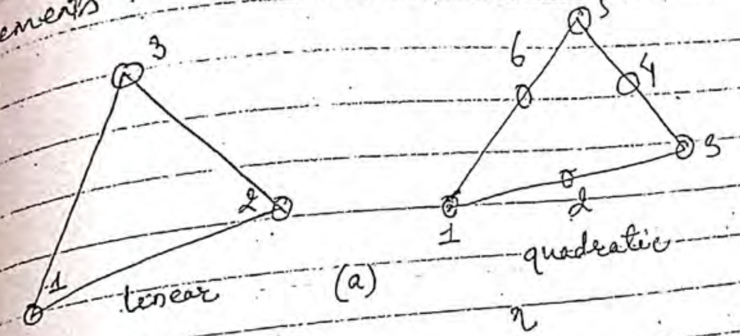
$x \quad y$ — ~~tin~~ linear

$x^2 \quad xy \quad y^2$ — Quadratic

$x^3 \quad x^2y \quad xy^2 \quad y^3$ — cubic

$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$ — Quartic are interpolated are called [illegible]

elements.



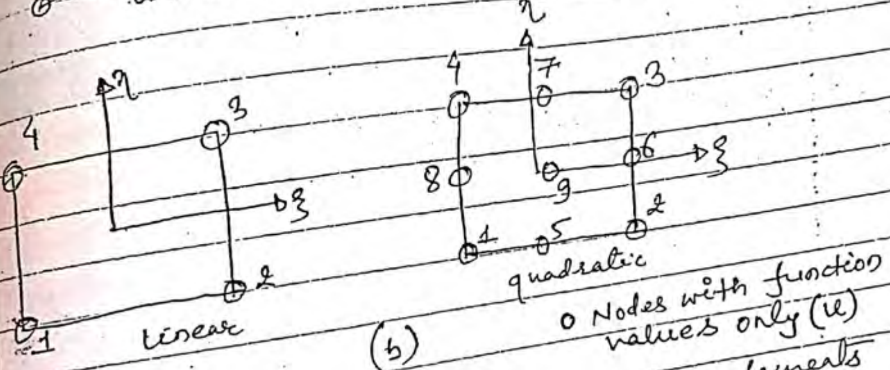


Fig (a) linear and quadratic triangular elements of Lagrangian family

fig (b) linear and quadratic rectangular elements of Langrange family.

Formulation of triangular elements (three noded) (CST) will be done in coming chapter.

Rectangular elements

→ The Lagrange family of rectangular elements can be developed from rectangular array as shown in fig. Since the linear rectangular element has four corners (hence four nodes), the polynomial should have the 1st four terms 1, x, y and xy (which form parallelogram in Pascal triangle and a ~~rectan~~ and a rectangle in array as shown in fig.

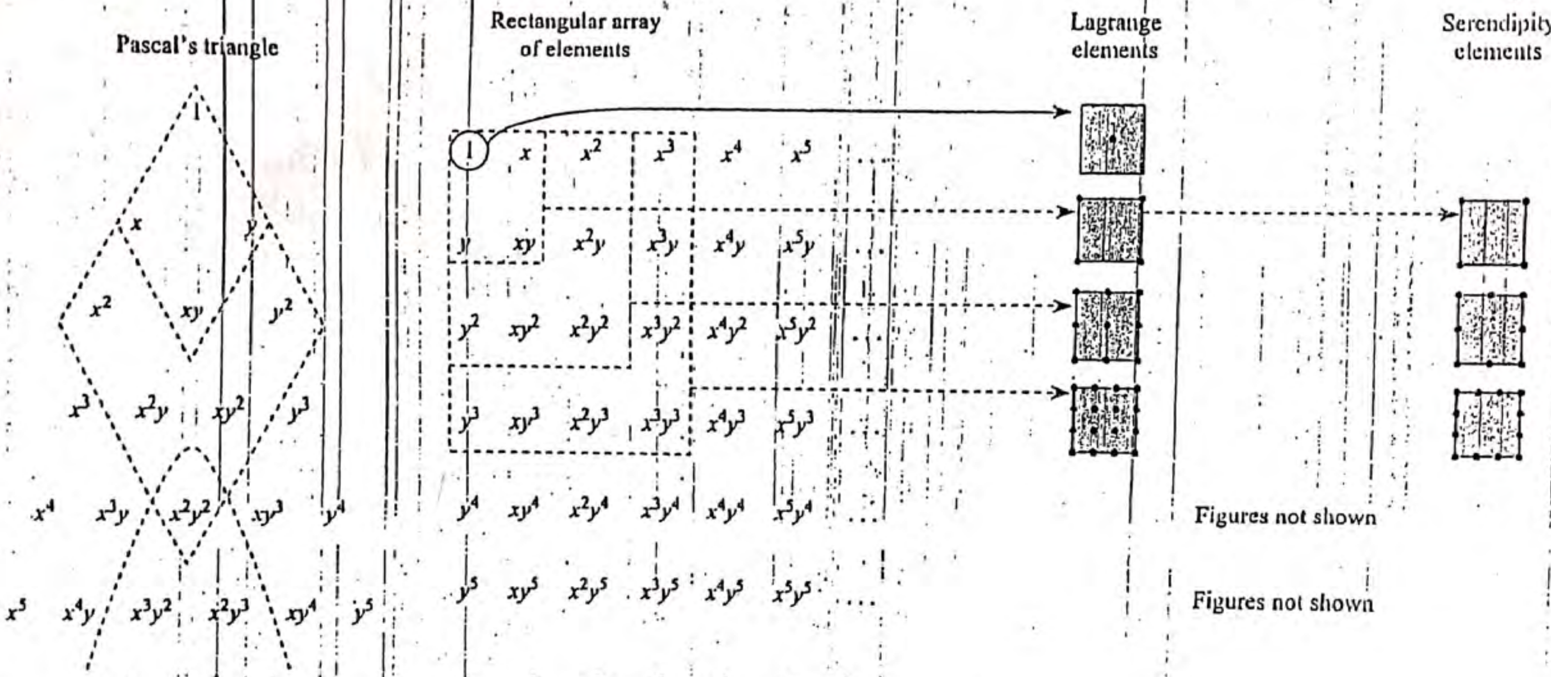


Figure 9.2.5 Lagrange and serendipity families of rectangular finite elements,

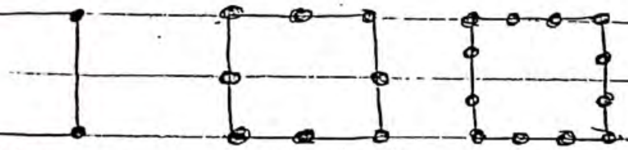
Hermite interpolation for beam element

Hermite family of interpolation function which polate the function and its derivatives.

beam element, it is discussed already.

Serendipidity element

pe Serendipidity elements are those rectangular ents which have no interior nodes. In other ls, all the nodes points are on the boundary the element.



linear   b) quaddratic   c) ~~linear~~ cubic c

Fig:- rectangular elements of Serendipity ~~element~~ family.

2) Rectangular element (Shaded linear only)

From chapter 3, We know,

$$\delta = \begin{Bmatrix} u \\ v \end{Bmatrix} \qquad \{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \qquad \varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

~~Where~~ $\{f\} = \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Triangular element (constant strain triangle)

- Two dimensional region is divided into straight-sided triangles as shown in fig.
- The points where the corner of the triangles meet are called nodes and each triangle formed by three nodes and three sides is called element.
- Each node is permitted to displace in two direction x and y. Thus each node has two degree of freedom.

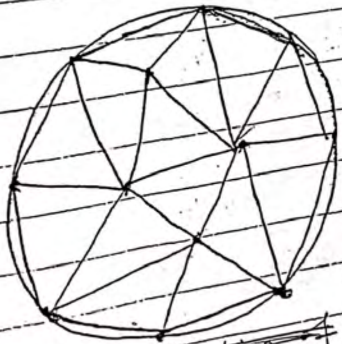


Fig:- ~~Representation~~ discretization of 2-D domain by triangles.

on x and 3 ~~eqn~~ on y.

displacement field on x and direction contains a polynomial containing three generalized ordinates. ~~plane~~

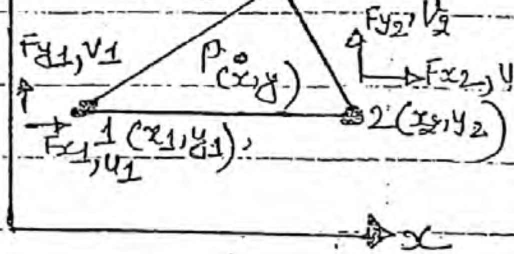


Fig 1 – Constant strain triangle.

$u = a_1 + a_2x + a_3y$ —⟨i⟩

$V = a_4 + a_5x + a_6y$ —⟨ii⟩

here, u = displacement at any point (x, y) in x dirⁿ

v = displacement at any point (x, y) in y dirⁿ

In matrix form,

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

$$\delta\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = [X]\{a\}$$ —⟨iii⟩

At nodes

$= a_1 + a_2x_1 + a_3y_1$

$= a_1 + a_2x_2 + a_3y_2$

$_3 = a_1 + a_2x_3 + a_3y_3$

$_1 = a_4 + a_5x_1 + a_6y_1$

$_2 = a_4 + a_5x_2 + a_6y_2$

$= a_4 + a_5x_3 + a_6y_3$

$$\left\{\begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix}\right\} \quad \begin{bmatrix} 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

$$\left\{\begin{matrix} \underline{u} \\ \underline{v} \end{matrix}\right\} = [A]\{a\}$$

$\rightarrow$ nodal displacements.

$$\{a\} = [A]^{-1}\left\{\begin{matrix} \underline{u} \\ \underline{v} \end{matrix}\right\} \quad \text{—(iv)}$$

$$= \frac{1}{2\Delta}\begin{bmatrix} t_1 & t_2 & t_3 & 0 & 0 & 0 \\ y_{23} & y_{31} & y_{12} & 0 & 0 & 0 \\ x_{32} & x_{13} & x_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & t_1 & t_2 & t_3 \\ 0 & 0 & 0 & y_{23} & y_{31} & y_{12} \\ 0 & 0 & 0 & x_{32} & x_{13} & x_{21} \end{bmatrix}\left\{\begin{matrix} \underline{u} \\ \underline{V} \end{matrix}\right\} \quad \text{—(v)}$$

Where Δ = area of triangle $= \dfrac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

$$\left.\begin{aligned} t_1 &= x_2y_3 - y_2x_3 \\ t_2 &= x_3y_1 - y_3x_1 \\ t_3 &= x_1y_2 - y_1x_2 \end{aligned}\right\} t_i = x_jy_k - y_jx_k$$

$$y_{jk} = y_j - y_k, \quad \begin{aligned} y_{23} &= y_2 - y_3 \\ y_{12} &= y_1 - y_2 \\ y_{31} &= y_3 - y_1 \end{aligned}$$

$$x_{ij} = x_i - x_j, \quad \begin{aligned} x_{32} &= x_3 - x_2 \\ x_{13} &= x_1 - x_3 \end{aligned}$$

$$\begin{Bmatrix}u\\v\end{Bmatrix} = \begin{bmatrix}1 & x & y & 0 & 0 & 0\\0 & 0 & 0 & 1 & x & y\end{bmatrix}[A]\begin{Bmatrix}\underline{u}\\\underline{v}\end{Bmatrix} \text{ —(vi)}$$

$$\begin{Bmatrix}u\\v\end{Bmatrix} = [N]\begin{Bmatrix}\underline{u}\\\underline{v}\end{Bmatrix}$$

$$[N] = \begin{bmatrix}1 & x & y & 0 & 0 & 0\\0 & 0 & 0 & 1 & x & y\end{bmatrix}[A]^{-1}$$ Shape function

$$[N] = [X][A]^{-1} \text{ —(vii)}$$

Strain.

We have, $\{\varepsilon\} = [B]\{\underline{u}\} = \begin{Bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\end{Bmatrix}$

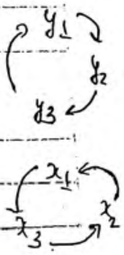
∴ (ε) From (vi)

$$\therefore \{\varepsilon\} = \begin{Bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial z}\end{Bmatrix} = \underbrace{\begin{bmatrix}0 & \underline{1} & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & \underline{1}\\0 & 0 & \underline{1} & 0 & \underline{1} & 0\end{bmatrix}[A]^{-1}}_{\Downarrow}\begin{Bmatrix}\underline{u}\\\underline{v}\end{Bmatrix}$$

$$\{\varepsilon\} = [B]\{\underline{u}\}$$

$$\begin{bmatrix} & & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} * \frac{1}{2\Delta} \begin{bmatrix} t_1 & t_2 & t_3 & 0 & 0 & 0 \\ y_{23} & y_{31} & y_{12} & 0 & 0 & 0 \\ x_{32} & x_{13} & x_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & t_1 & t_2 & t_3 \\ 0 & 0 & 0 & y_{23} & y_{31} & y_{12} \\ 0 & 0 & 0 & x_{32} & x_{13} & x_{21} \end{bmatrix}$$

$$[B] = \frac{1}{2\Delta} \begin{bmatrix} y_{23} & y_{31} & y_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{32} & x_{13} & x_{21} \\ x_{32} & x_{13} & x_{21} & y_{23} & y_{31} & y_{12} \end{bmatrix}$$



Hence, Matrix [B] is only function of nodal coordinate.

$$\{\varepsilon\} = [B]\{\underline{u}\}.$$

From this strain is function of nodal coordinates and nodal displacements which are constants. Therefore, strain is constant within the element. Hence this element called constant strain triangle element.

Stiffness matrix

$$\{\sigma\} = [D]\{\varepsilon\}$$

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu/2 \end{bmatrix}$$ For plane stress

$$= \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$ For plane strain

$$[K] = \int_V [B]^T[D][B]\,dv = [B]^T[D][B]\int_V dV = [B]^T[D][B]\cdot\Delta * t$$

$$= t\Delta[B]^T[D][B]$$

and, $\{\sigma\} = [D]\{\varepsilon\}$

$$\{\varepsilon\} = [B]\{\underline{u}\}$$

$$\therefore U_e = \frac{1}{2}\int_V \left[[D]\{\varepsilon\}\right]^T \{\varepsilon\} dv = \frac{1}{2}\int_V \{\varepsilon\}^T [D]^T \{\varepsilon\} dv$$

$$U_e = \frac{1}{2}\int_V \{\varepsilon\}^T [D]\{\varepsilon\}\, dv \qquad \because [D]^T = [D] \text{ (symmetric matrix)}$$

$$U_e = \frac{1}{2}\int_V \left[[B]\{\underline{u}\}\right]^T [D]\, [B]\{\underline{u}\}\, dv$$

$$U_e = \frac{1}{2}\int_V \{\underline{u}\}^T [B]^T [D][B]\{\underline{u}\}\, dv$$

Here, all matrices are constant

$$\therefore U_e = \frac{1}{2}\{\underline{u}\}^T [B]^T [D][B]\{\underline{u}\} \int_V dv$$

$$U_e = \frac{1}{2}\{\underline{u}\}^T [B]^T [D]\, [B]\{\underline{u}\}\; t * A$$

Where t = element thickness

A = Area of element

$$U_e = \frac{1}{2}\{\underline{u}\}^T tA[B]^T [D][B]\{\underline{u}\} \quad \text{— (i)}$$

Also, $U_e = \frac{1}{2}\{\underline{u}\}^T [K]\{\underline{u}\}$ —— (ii)

comparing (i) & (ii), we obtain:

$$\boxed{[K] = tA\, [B]^T [D]\, [B]}$$

When expressed in global coordinate system, the interpolation function or shape function for the triangular element are algebraically complex. Further, the integration required to obtain element characterstic matrices are cumbersome. The construction of interpolation or shape function as well as the subsequently required integration become much easier if we use natural coordinates or area coordinates or triangular coordinates system.

The three shape function N_1, N_2, & N_3 corresponding to nodes 1, 2, and 3, respectively are shown in fig.. For the constant strain triangle, the shape functions are linear over the element.

shape function N_1 is unity at node 1 and linearly reduces to 0 at nodes 2 and 3 as shown in fig (a)

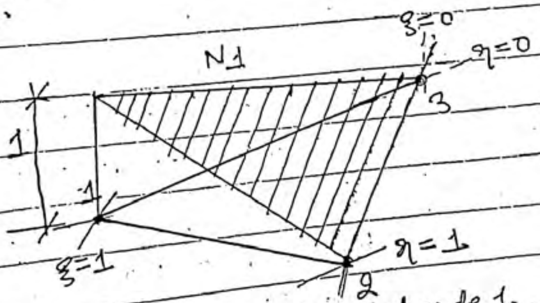


fig (a) shape function at node 1
N_1

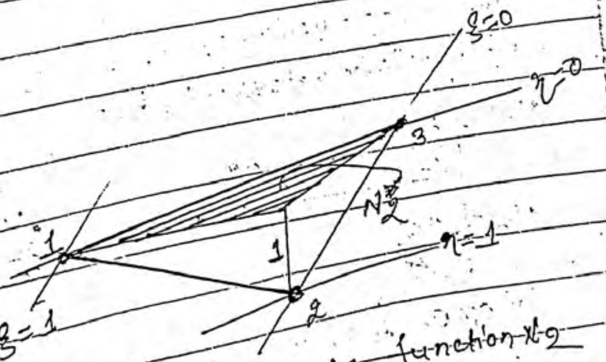
Here, at $\xi=0$, $N_1=0$.
linear relationship between N_1 and ξ,

$\Rightarrow N_1 = K\xi$

at $\xi=1$, $N_1=1$

$1 = K\cdot 1$, $\therefore K=1$

$\Rightarrow N_1 = \xi$



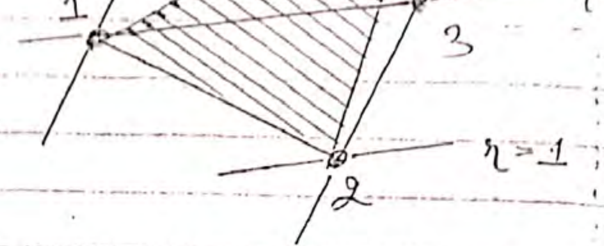
Fig(b) shape function N_2

At $\eta=0$, $N_2 = 0$

$N_2 = K\eta$

At $\eta=1$, $N_2=1$, then $K=1$

$N_2 = \eta$



Fig(c) Shape function N_3

In particular, $N_1 + N_2 + N_3$ represents a plane at height of 1 at nodes 1, 2, and 3 and thus plane is parallel to triangle 1 2 3.

Consequently,

$$N_1 + N_2 + N_3 = 1$$

$$\Rightarrow \xi + \eta + N_3 = 1$$

$$\Rightarrow N_3 = 1 - \xi - \eta$$

Where ξ, η are natural coordinates ~~(x and~~ (in x & y direction)

- The shape functions can be physically represented by area coordinates.

Following fig shows a three-node triangular element divided into three areas defined by the nodes and an arbitrary interior point P(x,y). Note: P is not a node.

The shape function N_1, N_2 and N_3 are precisely represented by

$N_1 = \frac{A_1}{A}$; $N_2 = \frac{A_2}{A}$

$N_3 = \frac{A_3}{A}$

Where A is the area of element.

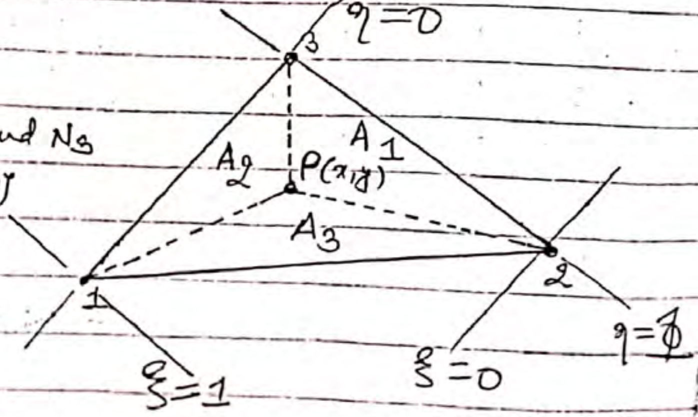


Fig:- Areas used to define area coordinates for triangular element consider a triangular element, with nodal displacement as shown in fig:

The displacements inside element are now written using the shape function and nodal values.

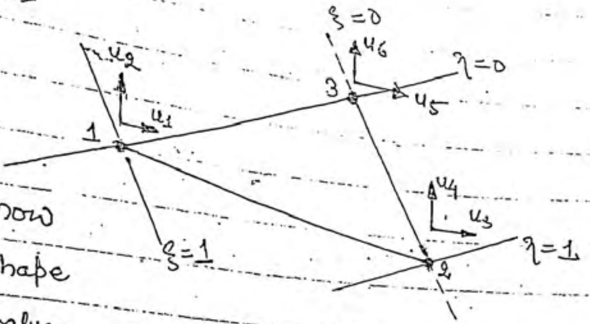


Fig:- nodal displacements notation for triangular element.

We have,

$$\left.\begin{aligned} u &= N_1u_1 + N_2u_3 + N_3u_5 \\ v &= N_1u_2 + N_2u_4 + N_3u_6 \end{aligned}\right\} \longrightarrow \langle i \rangle$$

or,

$$u = \xi u_1 + \eta u_3 + (1-\xi-\eta)u_5 = (u_1-u_5)\xi + (u_3-u_5)\eta + u_5$$

$$v = \xi u_2 + \eta u_4 + (1-\xi-\eta)u_6 = (u_2-u_6)\xi + (u_4-u_6)\eta + u_6$$

$\longrightarrow$ ⓘⓘ

Writing in matrix form, for eqn $\langle i \rangle$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \end{Bmatrix}$$

$$\{u\} = [N]\{\underline{u}\} \longrightarrow (iii)$$

$\hookrightarrow$ displacement at any point in the element

$\{\underline{u}\}$ = nodal displacement matrix

$[N]$ = shape function matrix

For triangular element, the coordinate x, y can also be in terms of nodal coordinates using same function. This is known as isoparametric representation:

$$\left.\begin{aligned} x &= N_1x_1 + N_2x_2 + N_3x_3 \\ y &= N_1y_1 + N_2y_2 + N_3y_3 \end{aligned}\right\} \langle iv \rangle$$

or,

$$x = \xi x_1 + \eta x_2 + (1-\xi-\eta)x_3 = (x_1-x_3)\xi + (x_2-x_3)\eta + x_3$$

$$y = \xi y_1 + \eta y_2 + (1-\xi-\eta)y_3 = (y_1-y_3)\xi + (y_2-y_3)\eta + y_3$$

$\longrightarrow$ Ⓥ

$$\left.\begin{aligned} x &= x_{13}\xi + x_{23}\eta + x_3 \\ y &= y_{13}\xi + y_{23}\eta + y_3 \end{aligned}\right\}$$

This equation relates x- and y-coordinates to the ξ- and η- coordinates.

Strain

Here, from⟨ii⟩, u and v are the function of ξ and η.

From ⟨vi⟩ x and y are the function of ξ and η.

Here, u and v are also the function of x and y.

$$\text{or, } u = u\Big(x(\xi,\eta),\, y(\xi,\eta)\Big) \;\&\; v = v\Big(x(\xi,\eta),\, y(\xi,\eta)\Big)$$

using chain rule;

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

In matrix form

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} \quad \text{——} \langle vii \rangle$$

Where (2*2) matrix is denoted as the Jacobian of transformation J.

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \quad \text{——} \langle viii \rangle$$

$\langle vi \rangle$,

$$[J] = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}$$

Also, from $\langle vii \rangle$,

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} \quad \text{——} \langle ix \rangle$$

and, $$[J]^{-1} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}^{-1} = \frac{1}{\det[J]} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}$$

$$\det[J] = x_{13}y_{23} - x_{23}y_{13} = 2 * \text{area of triangle}$$

From (ix) $$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\det[J]} \begin{Bmatrix} y_{23}\frac{\partial u}{\partial \xi} - y_{13}\frac{\partial u}{\partial \eta} \\ -x_{23}\frac{\partial u}{\partial \xi} + x_{13}\frac{\partial u}{\partial \eta} \end{Bmatrix} \quad \text{——} (x)$$

Replacing u by the displacement v, we get a similar expression

$$\begin{Bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\det[J]} \begin{Bmatrix} y_{23}\frac{\partial v}{\partial \xi} - y_{13}\frac{\partial v}{\partial \eta} \\ -x_{23}\frac{\partial v}{\partial \xi} + x_{13}\frac{\partial v}{\partial \eta} \end{Bmatrix} \quad \text{——} \langle xi \rangle$$

Now, From (ii), $\langle x \rangle$ and $\langle xi \rangle$

$$\{\varepsilon\} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{\det[J]} \begin{Bmatrix} y_{23}(u_1-u_5) - y_{13}(u_3-u_5) \\ -x_{23}(u_2-u_6) + x_{13}(u_4-u_6) \\ -x_{23}(u_1-u_5) + x_{13}(u_3-u_5) + y_{23}(u_2-u_6) - y_{13}(u_4-u_6) \end{Bmatrix}$$

$$= \frac{1}{\det[J]} \begin{Bmatrix} y_{23}u_1 - y_{23}u_5 - y_{13}u_3 + y_{13}u_5 \\ -x_{23}u_2 + x_{23}u_6 + x_{13}u_4 - x_{13}u_6 \\ -x_{23}u_1 + x_{23}u_5 + x_{13}u_3 - x_{13}u_5 + y_{23}u_2 - y_{23}u_6 - y_{13}u_4 + y_{13}u_6 \end{Bmatrix}$$

Here, $y_{13} = -y_{31}$ & $y_{12} = y_{13} - y_{23}$

$$= \frac{1}{\det[J]} \begin{Bmatrix} y_{23}u_1 + y_{31}u_3 + y_{12}u_5 \\ x_{32}u_2 + x_{13}u_4 + x_{21}u_6 \\ x_{32}u_1 + y_{23}u_2 + x_{13}u_3 + y_{31}u_4 + x_{21}u_5 + y_{12}u_6 \end{Bmatrix}$$

This eq^n can be written in the form,

$$\{\epsilon\} = [B]\{u\}$$

$$[B] = \frac{1}{\det[J]} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{32} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

Force vector

body force $\{f^e\} = \int_V [N]^T \{f\}\, dV$ $\{f\}$ = body force per unit volume

$$= \int_V [N] \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} dV$$

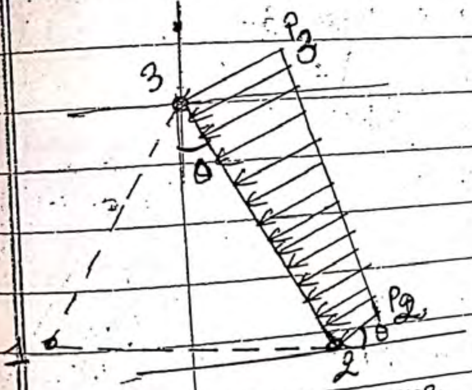
Traction force is distributed load acting on the surface of body.

$$\{T^e\} = \int [N]^T \{T\}\, ds = \int [N]^T \begin{Bmatrix} T_x \\ T_y \end{Bmatrix} ds$$

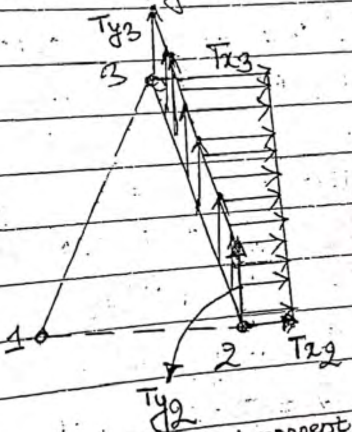
FORCE MATRIX

① Traction force (Distributed force)

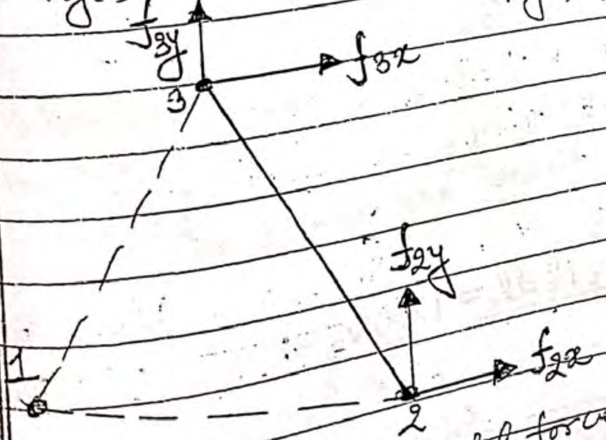
Frequently, the boundary condition for structural problems involve distributed loading on some portion of the geometric boundary. Such loading may arise from applied pressure (normal stress) or shearing loads. Consider following ~~fig~~ CST ~~as t.~~ element in which normal pressure is distributed along edge 2-3 as shown in fig(a)



Fig(a) Normal pressure



Fig(b) ~~press~~ component distribution



Fig(c) equivalent nodal forces due to distributed load.

Element thickness is denoted by t and the loads are assumed to be expressed in terms of force per unit area. We seek to replace the distributed loads with equivalent forces acting at nodes 2 and 3.

Considering traction force term from potential energy eqn.

$$\int_L \{u\}^T [T]\, t\, dl = \int_{l_{2-3}} \{u \quad v\} \begin{Bmatrix} T_x \\ T_y \end{Bmatrix} t\, dl$$

~~We are considering only 2-3.~~

$$= \int_{l_{2-3}} \left(u T_x + v T_y \right) t\, dl \qquad \text{——} \langle i \rangle$$

We are considering only edge 2-3.

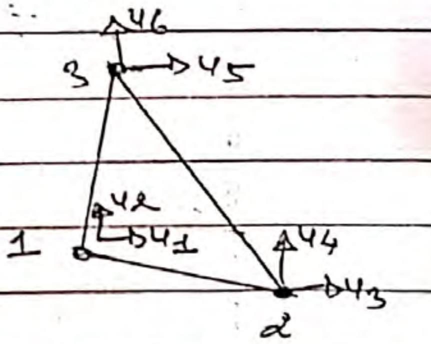


Fig:- nodal displacement notation

using interpolation relation,

$$u = N_2 u_3 + N_3 u_5$$

$$v = N_2 u_4 + N_3 u_6 \qquad \text{——} \text{(ii)}$$

$$T_x = N_2 T_{x2} + N_3 T_{x3}$$

$$T_y = N_2 T_{y2} + N_3 T_{y3}$$

substituting these in above eqn.

$$\Rightarrow \int_{l_{2-3}} \left(u T_x + v T_y \right) t\, dl = \int_{l_{2-3}} \left\{ \begin{array}{l} (N_2 u_3 + N_3 u_5)(N_2 T_{x2} + N_3 T_{x3}) \\ + (N_2 u_4 + N_3 v_6)(N_2 T_{y2} + N_3 T_{y3}) \end{array} \right\} t\, dl$$

$$\Big(N_2^2 u_3 Tx_2 + N_2N_3u_3Tx_3 + N_2N_3u_5Tx_2 + N_3^2u_5Tx_3 + N_2^2u_4Ty_2$$

$$+ N_2N_3u_4Ty_3 + N_2N_3u_6Ty_2 + N_3^2u_6Ty_3\Big)\,dl \quad (iii)$$

$_{2\text{-}3}$

Here displacements and forces are constant so integration only involves in shape function and

ng that,

$$N_2^2\,dl = \frac{1}{3}l_{2\text{-}3} \qquad \int_{2\text{-}3} N_3^2\,dl = \frac{1}{3}l_{2\text{-}3}$$

$_{3}$

$$\int_{l_{2\text{-}3}} N_2N_3\,dl = \frac{1}{6}l_{2\text{-}3}$$

, eq^n ⓘⓘⓘ becomes,

Let, $l_{2\text{-}3} = l$

$$\frac{l}{3}u_3Tx_2 + \frac{l}{6}u_3Tx_3 + \frac{l}{6}u_5Tx_2 + \frac{l}{3}u_5Tx_3 + \frac{l}{3}u_4Ty_2 + \frac{l}{6}u_4Ty_3$$

$$+ \frac{l}{6}u_6Ty_2 + \frac{l}{3}u_6Ty_3\Big)$$

$$\frac{tl}{6}\left(2Tx_2 + Tx_3\right)u_3 + \frac{tl}{6}\left(2Ty_2 + Ty_3\right)u_4 + \frac{tl}{6}\left(Tx_2 + 2Tx_3\right)$$

$$+ \frac{tl}{6}\left(Ty_2 + 2Ty_3\right)$$

$$\int_{l_{2\text{-}3}} \{u\}^T\{T\}\,t\,dl = \begin{Bmatrix} u_3 & u_4 & u_5 & u_6 \end{Bmatrix} \frac{tl}{6} \begin{Bmatrix} 2Tx_2 + Tx_3 \\ 2Ty_2 + Ty_3 \\ Tx_2 + 2Tx_3 \\ Ty_2 + 2Ty_3 \end{Bmatrix}$$

$$u_5 \quad u_6\}\{T\}$$

$$\{u_3 \quad u_4$$

Traction force matrix is given

$$[T] = \frac{tl}{6}\begin{Bmatrix} 2T_{x_2} + T_{x_3} \\ 2T_{y_2} + T_{y_3} \\ T_{x_2} + 2T_{x_3} \\ T_{y_2} + 2T_{y_3} \end{Bmatrix} = \begin{Bmatrix} f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{Bmatrix}$$

~~Where,~~ [illegible] From fig (a) & (b)

$$T_{x_2} = -P_2 \cos\theta \,,\quad T_{y_2} = -P_2 \sin\theta$$

$$T_{x_3} = -P_3 \cos\theta \quad T_{y_3} = -P_3 \sin\theta$$

In general for node i, j

$$[T] = \frac{t\,l_{i-j}}{6}\begin{Bmatrix} 2T_{x_i} + T_{x_j} \\ 2T_{y_i} + T_{y_j} \\ T_{x_i} + 2T_{x_j} \\ T_{y_i} + 2T_{y_j} \end{Bmatrix} = \begin{Bmatrix} f_{ix} \\ f_{iy} \\ f_{jx} \\ f_{jy} \end{Bmatrix}$$

general, a body force is noncontact force
ing on a body on a per unit mass basis. The
t commonly encountered ~~force~~ body forces are
vitational attraction (weight), centrifugal forces
sing from rotational motion and magenetic
rce.

The body force term from potential energy
n.

$$\int_V \{u\}^T\{f\} V = \int_A \{u \quad v\}\begin{Bmatrix} f_x \\ f_y \end{Bmatrix} t\,dA$$

$$= t\int_A \left(u f_x + v f_y\right) dA$$

Where f_x and f_y are the force per unit volume in respective coordinate dirn.

using interpolation relation:

$$\int \{u\}^T\{f\}\, t\,dA = t\int \left\{ \left(N_1u_1 + N_2u_3 + N_3u_5\right) f_x + \left(N_1u_2 + N_2u_4 + N_3u_6\right) f_y \right\} dA$$

$$= \left(t f_x \int_e N_1\,dA\right)u_1 + \left(t f_y \int_e N_1\,dA\right)u_2 + \left(t f_x \int N_2\,dA\right)u_3 + \left(t f_y \int N_2\,dA\right)u_4 + \left(t f_x \int N_3\,dA\right)u_5 + \left(t f_y \int N_3\,dA\right)u_6$$

noting that,

$$\int N_1\, dA = \frac{A}{3} \qquad \int N_1^2\, dA = \frac{A}{6} \qquad \int N_1 N_2 N_3\, dA = \frac{\Delta}{60}$$

$$\int N_2\, dA = \frac{A}{3} \qquad \int N_3\, dA = \frac{A}{3}$$

Now, above equation becomes,

$$\int_e \{u\}^T \{f\}\, t\, dA = \left(\frac{tA}{3} f_x\right) u_1 + \left(\frac{tA}{3} f_y\right) u_2 + \left(\frac{tA}{3} f_x\right) u_3 + \left(\frac{tA}{3} f_y\right) u_4 + \left(\frac{tA}{3} f_x\right) u_5 + \left(\frac{tA}{3} f_y\right) u_6$$

$$= \{u_1 \; u_2 \; u_3 \; u_4 \; u_5 \; u_6\} \frac{tA}{3} \begin{Bmatrix} f_x \\ f_y \\ f_x \\ f_y \\ f_x \\ f_y \end{Bmatrix}$$

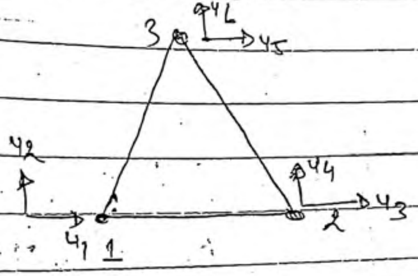
$$= \{u\}\{f\}$$

Where $\{f\}$ = body force matrix

$$= \frac{tA}{3} \begin{Bmatrix} f_x \\ f_y \\ f_x \\ f_y \\ f_x \\ f_y \end{Bmatrix}$$

Body force from Natural coordinate system

① compute the consistent load vector when the self weight of ρ/unit volume is acting in a constant strain triangle element (ρ is acting in negative y direction).



Sol^n let t = thickness

Consistent load vector is given by

$$\{\} = \int_V [N]^T\{f\}dV + \underbrace{\int_S [N]^T\{T\}ds}_{\text{(Traction force)}} \rightarrow 0$$

Since there is no distributed force on CST.

$$\{F\} = \int_V [N]^T\{f\}dv = t\int_A [N]^T\{f\}dA$$

$$\{F\} = t\int_A \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} dA$$

p.t.o

$$\{F\} = t\int_A \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{Bmatrix} 0 \\ -\rho \end{Bmatrix} dA, \quad f_x = 0$$

∵ there is no weight in x dirⁿ

$$= t\int_A \begin{Bmatrix} 0 \\ -N_1\rho \\ 0 \\ -N_2\rho \\ 0 \\ -N_3\rho \end{Bmatrix} dA$$

Now we have.

$$I = \int_A x^m y^n z^p \, dA = \frac{2\Delta \; m!\, n!\, p!}{(m+n+p+2)!}$$

$$\text{Now, } \int_A N_1 \, dA = \int_A N_1^1 N_2^0 N_3^0 \, dA = \frac{2\Delta \times 1!}{3!} = \frac{\Delta}{3} \leftarrow \text{Area of CST}$$

$$\int_A N_2 \, dA = \int_A N_1^0 N_2^1 N_3^0 \, dA = \frac{\Delta}{3}$$

$$\int_A N_3 \, dA = \frac{\Delta}{3}. \quad \text{Hence } \{F\} = t\begin{Bmatrix} 0 \\ -\Delta\rho/3 \\ 0 \\ -\Delta\rho/3 \\ 0 \\ -\Delta\rho/3 \end{Bmatrix} = \frac{-t\Delta\rho}{3}\begin{Bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{Bmatrix}$$

ULATION

method is a powerful technique for analysing engineering problems involving complex, irregular geometries. It is to be noted that there are several draw-backs in attempting to represent a curved line as well as a body with curved boundaries by straight lines as well as straight sided elements. A large number of elements may be necessary to obtain a resonable resemblance between original body and the assemblage. ~~One way of approaching the problem~~ An accurate representation of irregular domains (geometry) (i.e. domain with curved boundaries) can be done by the use of refined meshes and/or irregularly shaped curvilinear elements. Isoparametric representation is the most efficient in this respect.

Same interpolation function or shape function are used for both the variation of unknown displacent field and description of element geometry, the procedure is known as Isoparametric mapping. The element defined by such procedure is known as isoparametric element.

For eg. The four-node quadrilateral element is derived from four node rectangular element (known as parent element) via mapping process.

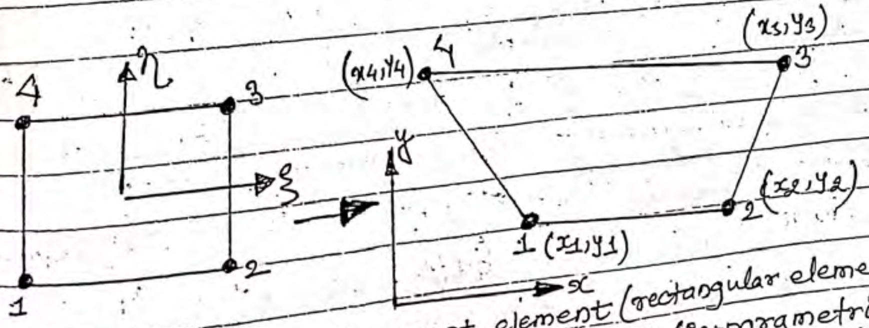


Fig 1:- Mapping of parent element (rectangular element) into an quadrilateral element (isoparametric element)

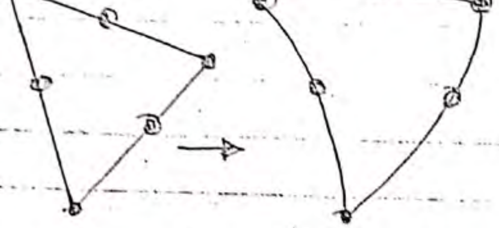


Fig :- Isoparametric mapping of quadratic ~~elements into curved elements~~ triangular element into curved element

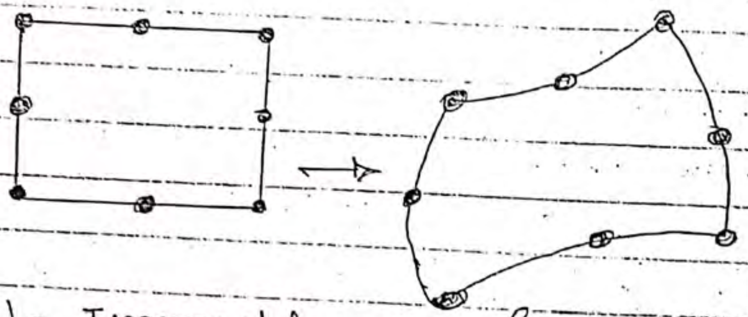


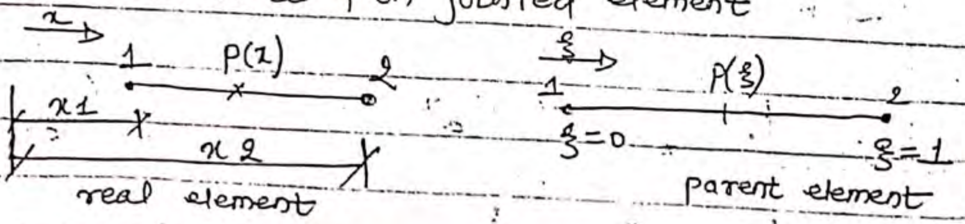
Fig :- Isoparametric mapping of quadratic rectangular element into curved element

There are two essential ingredients needed for representing the curved elements,

1) Natural coordinate system
2) Interpolation displacement model

Basic principle of shape function mapping

Ex Consider a pin jointed element

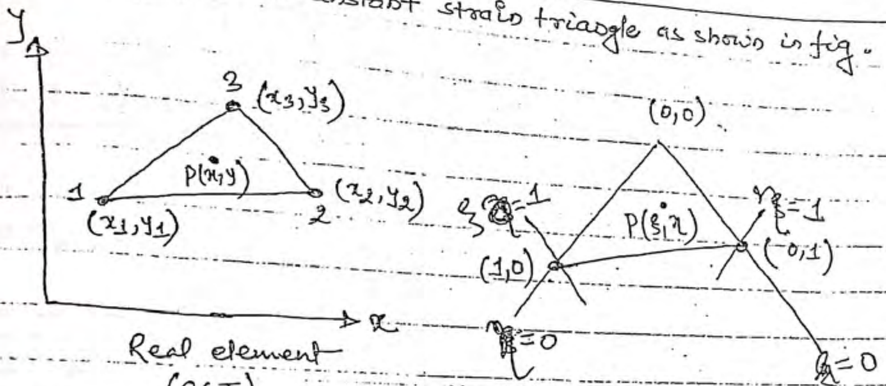


For every point P in parent, there is corresponding point in the real element.

$$x = [N_1 \quad N_2] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$x = [N]\{\underline{x}\}$, $N_1 = \frac{1-\xi}{2}$ $N_2 = \underline{1+\xi}$

let us consider a constant strain triangle as shown in fig.



Real element (CST)

parent element (CST) (master element)

Consider any point p in real element which is mapping of P(ξ,η) from parent element.

1st develop the shape functions on a ~~master elemen~~ parent element (master element). The parent element is defined in ξ-η coordinates (or natural coordinate system)

We have already derived that,

$N_1 = \xi, \quad N_2 = \eta$ & $N_3 = 1-\xi-\eta$

The transformation between (x,y) and (ξ,η) is accomplished by a coordinate transform of the form

$$x = \sum_{j=1}^{m} x_j N_j(\xi,\eta) \qquad y = \sum_{j=1}^{m} y_j N_j(\xi,\eta) \quad \text{—} \langle i \rangle$$

Where, j = 1,2, —— (node number) (Here m=3)

From ⓘ For triangular element

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

In matrix form

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \end{Bmatrix}$$

$\left. y_5 \right]$

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = [N] \begin{Bmatrix} \underline{x} \\ \underline{y} \end{Bmatrix}$$

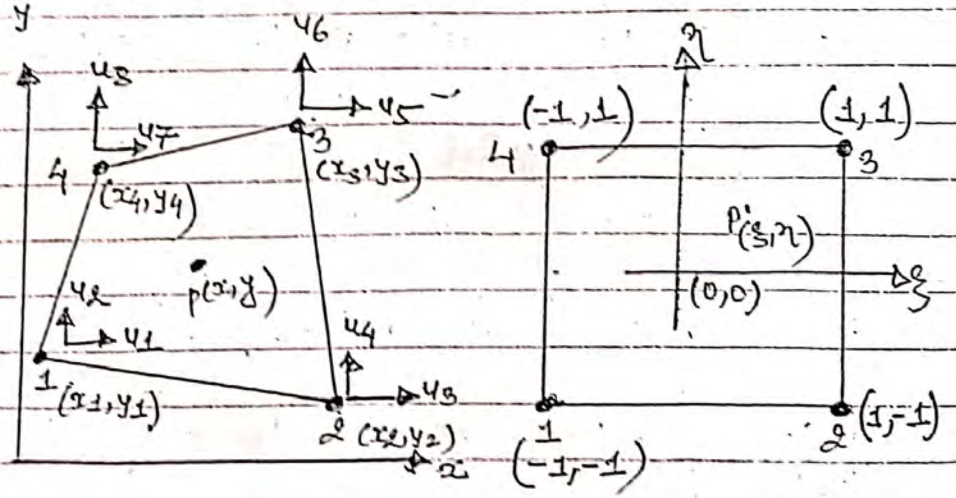
↳ coordinate of any point P

$[N]$ = mapping function

$\begin{Bmatrix} \underline{x} \\ \underline{y} \end{Bmatrix}$ = nodal coordinate

Linear Quadrilateral (Four noded)

Consider any ~~axibr~~ arbitrary quadrilateral element in cartesian coordinate and the corresponding parent element in ~~local coordinate sy~~ natural coordinate system. Since, quadrilateral element is derived from rectangular element, hence rectangular element is parent element.



real element.   parent element shape function for parent element

We know that, shape function N_i is unity at node i and zero at other nodes.

$N_1 = 1$ at node 1
$\quad = 0$ at nodes 2, 3, & 4

Requirement that $N_1=0$ at nodes 2, 3, and 4 is equivalent that $N_1 = 0$, along edges $\xi = +1$ and $\eta = +1$. Thus N_1 has to be of the form

$$N_1 = K(1-\xi)(1-\eta) \quad \text{——} \langle i \rangle$$

Where K is some constant.

At, $\xi = -1;\ \eta = -1$ (i.e. node 1), $N_1 = 1$, substituting in Eq $\langle i \rangle$

$$1 = K(1--1)(1--1) \quad \therefore K = 1/4$$

Thus, $N_1 = \frac{1}{4}(1-\xi)(1-\eta)$

For N_2.

N_2 is 1 at node 2.
$\quad = 0$ at 1, 3, 4,

$$N_2 = K(1+\xi)(1-\eta)$$

At $\xi = 1;\ \eta = -1$, $N_2 = 1$,

$$1 = K(1+1)(1+1) \quad \therefore K = 1/4$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

Similarly,

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

We express the displacement field within the elements in terms of nodal values.

$$u = N_1u_1 + N_2u_3 + N_3u_5 + N_4u_7$$

$$v = N_1u_2 + N_2u_4 + N_3u_6 + N_4u_8$$

In matrix form

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_8 \end{Bmatrix}$$

$$\{u\} = [N]\{\underline{u}\} \longrightarrow \langle ii \rangle$$

In isoparametric formulation same shape function is used.

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$

$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_4 \\ y_1 \\ \vdots \\ y_4 \end{Bmatrix}$$

OR

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{Bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_4 \\ y_4 \end{Bmatrix}$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = [N] \begin{Bmatrix} \underline{x} \\ \underline{y} \end{Bmatrix} \longrightarrow \langle iii \rangle$$

https://civinnovate.com/civil-engineering-notes/

$$J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = ~~\frac{\partial N_1}{\partial \xi} \quad \frac{\partial N_2}{\partial \xi}~~$$

$$= \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

Displacement approximation

$$\{u\} = [N]\{\underline{u}\}$$

strain displacement relationship

$$= \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix}$$

$$\begin{matrix} v \\ x \\ v \\ y \end{matrix}\Bigg\} = [J]^{-1} \begin{Bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

$$\therefore \{\varepsilon\} = \underset{3*4}{\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&1&1&0\end{bmatrix}} \underset{(4*4)}{\begin{bmatrix}[J]^{-1} & [0]\\ [0] & [J]^{-1}\end{bmatrix}} \begin{Bmatrix} \frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \eta}\\ \frac{\partial v}{\partial \xi}\\ \frac{\partial v}{\partial \eta}\end{Bmatrix}_{(4*1)}$$

But

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \eta}\\ \frac{\partial v}{\partial \xi}\\ \frac{\partial v}{\partial \eta}\end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} & \langle 0 \rangle \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} & \langle 0 \rangle \\ \langle 0 \rangle & \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \langle 0 \rangle & \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \underline{u} \\ \underline{v} \end{Bmatrix}$$

$$= \begin{bmatrix} [B_n] & [0] \\ [0] & [B_n] \end{bmatrix} \begin{Bmatrix} \underline{u} \\ \underline{v} \end{Bmatrix}$$

Where, $[B_n] = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} [J] & [0] \\ [0] & [J]^{-1} \end{bmatrix} \begin{bmatrix} [B_\eta] & [0] \\ [0] & [B_\eta] \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$\{\varepsilon\} = [B] \begin{Bmatrix} u \\ v \end{Bmatrix}$$

Here

$$[B] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} [J]^{-1} & [0] \\ [0] & [J]^{-1} \end{bmatrix} \begin{bmatrix} [B_\eta] & [0] \\ [0] & [B_\eta] \end{bmatrix}$$

ELEMENT STIFFNESS MATRIX

$$[K] = \int_V [B]^T [D] [B] \, dv = t \int_A [B]^T [D] [B] \, dx \, dy$$

$$= t \int_{-1}^{1} \int_{-1}^{1} [B]^T [D] [B] \, d\xi \, d\eta \, \det [J]$$

Consistent load vector

$$[F] = \underbrace{t \int_{-1}^{1} \int_{-1}^{1} [N]^T \det[J] \, d\xi \, d\eta \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}}_{\text{body force}} + \underbrace{t \int_S [N]^T \begin{Bmatrix} T_x \\ T_y \end{Bmatrix} ds}_{\text{Traction force}}$$

Problems

① The nodal coordinates of the triangular element are shown in fig:-- At the interior point P, the x-coordinate is 3.3 and $N_1 = 0.3$. Determine N_2, N_3 and y coordinate at point P.

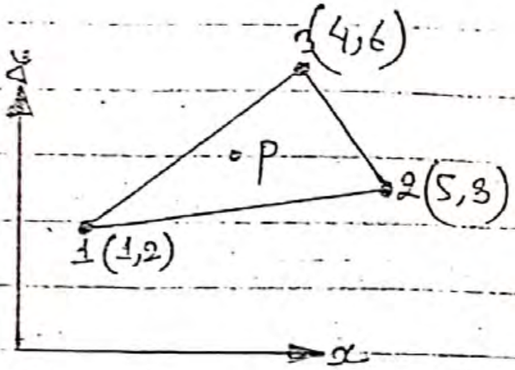


Fig p.1

Sol^n We know,

$N_1 = \xi, \; N_2 = \eta, \; N_3 = 1-\xi-\eta$

using isoparametric representation for triangular element,

$x = N_1 x_1 + N_2 x_2 + N_3 x_3$

$3.3 = 0.3 * 1 + \eta * 5 + (1 - 0.3 - \eta) 4$

$3.3 = 0.3 + 5\eta + 4 - 1.2 - 4\eta$

$\eta = 0.2 = N_2$ Ans

$N_3 = 1 - \xi - \eta = 1 - 0.3 - 0.2 = 0.5$ Ans

And $y = N_1 y_1 + N_2 y_2 + N_3 y_3 = 0.3 * 2 + 0.2 * 3 + 0.5 * 6$

$= 4.2$ //

2 Given the triangular ~~element~~ plane stress element shown in fig. Determine the nodal forces equivalent to the distributed load. Element thickness = 0.2 in and uniform.

Soln

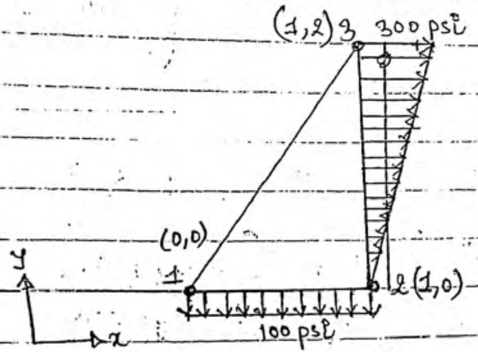
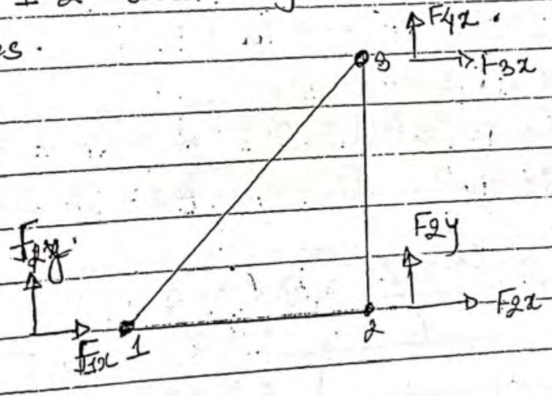


Fig 1.① Distributed loads on triangular element

Soln Here edge 1-2 and edge 2-3 are subjected to surface forces.



For edge 1-2

We have already derived that, equivalent nodal traction force $[T] = \frac{tl}{6}\begin{Bmatrix} 2Tx_1 + Tx_2 \\ 2Ty_1 + Ty_2 \\ Tx_1 + 2Tx_2 \\ Ty_1 + 2Ty_2 \end{Bmatrix}$

Here, surface forces are acted on global x-y ~~do~~ coordinate dirn.

So, $Tx_1 = 0$, $Tx_2 = 0$

$Ty_1 = -100$ psi $Ty_2 = -100$ psi

$$T^1 = \frac{0.2*1}{6}\begin{Bmatrix} 0+0 \\ -200-100 \\ 0+0 \\ -100-200 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10 \\ 0 \\ -10 \end{Bmatrix}\begin{array}{l} \text{Global Dof} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

For edg 2-3.

$$[T]^2 = \frac{t\, l_{2-3}}{6}\begin{Bmatrix} 2Tx_2 + Tx_3 \\ 2Ty_2 + Ty_3 \\ Tx_2 + 2Tx_3 \\ Ty_2 + 2Ty_3 \end{Bmatrix}$$

$l_{2-3} = 2$

Here, $P_2 = 0$ and $P_3 = 300$ psi

$Tx_2 = P_2 \cos\theta = 0,$

$Tx_3 = P_3 \cos\theta = 300 \cos 0 = 300$ psi

$Ty_2 = 0 \qquad Ty_3 = P_3 \sin\theta = 300 \sin 0 = 0$

$$[T]^2 = \frac{0.2*2}{6}\begin{Bmatrix} 0+300 \\ 0+0 \\ 0+2*300 \\ 0+0 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 0 \\ 40 \\ 0 \end{Bmatrix}\begin{array}{l} \text{Global Dof} \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

On Assembling $[T]^1$ and $[T]^2$

$$[T] = \begin{Bmatrix} 0 \\ -10 \\ 20 \\ -10 \\ 40 \\ 0 \end{Bmatrix} \therefore \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10 \\ 20 \\ -10 \\ 40 \\ 0 \end{Bmatrix}$$

Problem 8 -

acti

Consi

Here

For e

[T]

Here

Tx

Ty

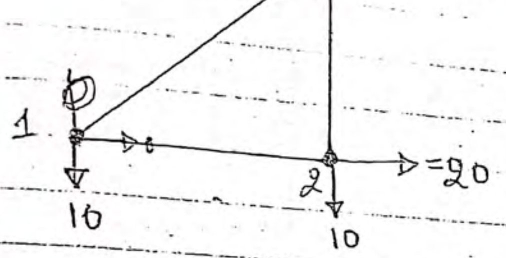
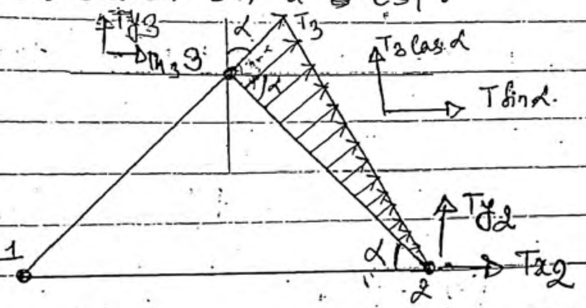


Fig !- equivalent nodal forces due to given distributed force.

blem 3. Compute consistent load vector when surface forces acting on side 23 as shown in a CST.



Consistent load vector = body force vector + traction force vector.

Here, body force ~~vector~~ is ~~are~~ neglected, so,

For edg 2-3

$$[T] = \frac{t\, l_{2-3}}{6} \begin{Bmatrix} 2Tx_2 + Tx_3 \\ 2Ty_2 + Ty_3 \\ Tx_2 + 2Tx_3 \\ Ty_2 + 2Ty_3 \end{Bmatrix}$$

Here, $P_2 = 0$, & $P_3 = T_3$

$Tx_2 = P_2 \sin\alpha = 0$, $Tx_3 = T_3 \sin\alpha$

$Ty_2 = 0$, $Ty_3 = T_3 \cos\alpha$

$$[T] = \frac{t\,l_{2-3}}{6}\begin{Bmatrix} 0 + T_3 \sin\alpha \\ 0 + T_3 \cos\alpha \\ 0 + 2T_3 \sin\alpha \\ 0 + 2T_3 \cos\alpha \end{Bmatrix}$$

$$[T] = \begin{Bmatrix} F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = T_3\,t\,l_{2-3} \begin{Bmatrix} \cancel{T_3}\,\frac{1}{6}\sin\alpha \\ \frac{1}{6}\cos\alpha \\ \frac{1}{3}\sin\alpha \\ \frac{1}{3}\cos\alpha \end{Bmatrix}$$

there is no distributed force on edge 1-2,

$\therefore$ $F_{1x} = 0$

$F_{1y} = 0$

$$[T] = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = T_3\,t\,l_{2-3} \begin{Bmatrix} 0 \\ 0 \\ \frac{1}{6}\sin\alpha \\ \frac{1}{6}\cos\alpha \\ \frac{1}{3}\sin\alpha \\ \frac{1}{3}\cos\alpha \end{Bmatrix}$$

SN 0.3

We

$= t$

$= t$

$= t l_2$

Now, $[T]$

$T_x = N$

$\theta =$

$T_x =$

Ily j

$T_y =$

Then $\{T$

$= t l_2$

Q.No.3 derivation by ~~local coordinate~~ natural coordinate system,

We have element traction force

$$\{T^e\} = \int_s [N]^T [T]\, ds$$

$$= \int_l [N]^T [T]\, dl \times t$$

$$= t \int_l [N]^T [T]\, dl$$

$$= t \int_0^{l_2} [N]^T [T]\, dl$$

$$= t \int_0^1 [N]^T [T]\, l_2\, d\eta \quad \text{For edg 2-3}$$

η varies 0 – 1

$$= t l_2 \int_0^1 [N]^T [T]\, d\eta$$

Now, $[T] = \begin{Bmatrix} T_x \\ T_y \end{Bmatrix}$

$$T_x = \cancelto{0}{N_1 Tx_1} + N_2 Tx_2 + Tx_3 N_3$$

$$= \eta Tx_2 + (1-\eta) Tx_3$$

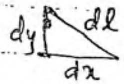
$Tx_2 = P_2 \sin\alpha = 0$, $Tx_3 = T_3 \sin\alpha$

$$T_x = (1-\eta) T_3 \sin\alpha$$

lly;

$$T_y = (1-\eta) T_3 \cos\alpha$$

Then $\{T^e\} = t l_2 \int_0^1 [N]^T \begin{Bmatrix} T_x \\ T_y \end{Bmatrix} d\eta$



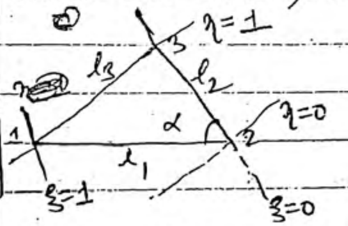
$$dl = \sqrt{dx^2 + dy^2}$$

$x = N_1 x_1 + N_2 x_2 + N_3 x_3$

And, $N_1 = \xi$, $N_2 = \eta$

$N_3 = 1 - \eta - \xi$

So, $x = \xi x_1 + \eta x_2 + (1 - \xi - \eta) x_3$



For line 2-3

$\xi = 0$;

$\therefore x = \eta x_2 + (1-\eta) x_3$

$= \eta (x_2 - x_3) + x_3$

$$\frac{dx}{d\eta} = x_2 - x_3$$

$dx = (x_2 - x_3) d\eta$

And $dy = (y_2 - y_3) d\eta$

Then $dl = \sqrt{((x_2-x_3)d\eta)^2 + ((y_2-y_3)d\eta)^2}$

$dl = d\eta\, l$

$\therefore dl = l\, d\eta$

$$= t l_2 \int_0^1 \begin{bmatrix} \cancelto{0}{N_1} & 0 \\ 0 & \cancelto{0}{N_1} \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{Bmatrix} (1-\eta) T_3 \sin\alpha \\ (1-\eta) T_3 \cos\alpha \end{Bmatrix} d\eta$$

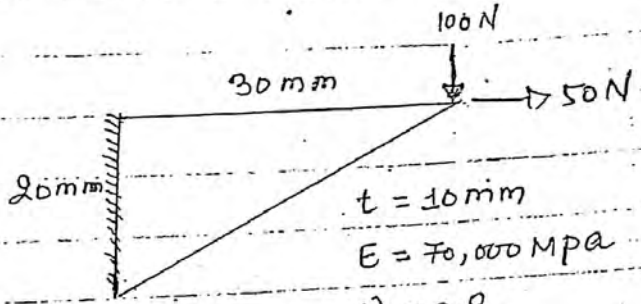
$$\{T^e\} = l_2 t \int_0 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \eta & 0 \\ 0 & \eta \\ 1-\eta & 0 \\ 0 & 1-\eta \end{bmatrix} \begin{Bmatrix} (1-\eta) T_3 \sin\alpha \\ (1-\eta) T_3 \cos\alpha \end{Bmatrix} d\eta$$

$$= T_3 \cdot l_2 t \int_0^1 \begin{Bmatrix} 0 \\ 0 \\ (\eta-\eta^2)\sin\alpha \\ (\eta-\eta^2)\cos\alpha \\ (1-\eta)^2 \sin\alpha \\ (1-\eta)^2 \cos\alpha \end{Bmatrix} d\eta$$

$$\{T^e\} = T_3 l_2 t \begin{Bmatrix} 0 \\ 0 \\ 1/6 \sin\alpha \\ 1/6 \cos\alpha \\ 1/3 \sin\alpha \\ 1/3 \cos\alpha \end{Bmatrix}$$

deflection at the point of load application using

one - element model.



t = 10 mm

E = 70,000 Mpa

$\nu = 0.3$

Fig ③

Sol^n

Assuming plane stress condition.

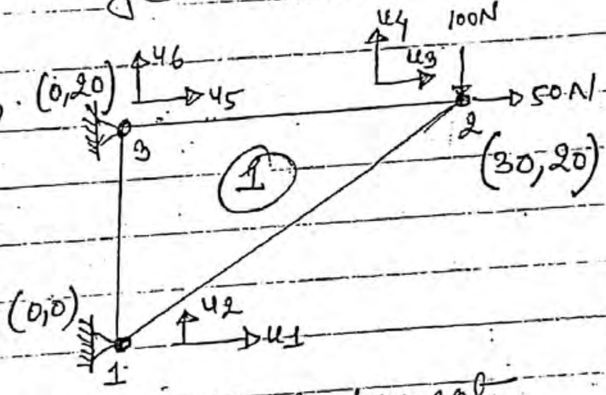


Fig :- nodal displacement notation

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Here, $E = 70,000 \text{ Mpa} = 70,000 * 10^6 \, N/m^2 = 70,000 * 10^6 * 10^{-6} \, N/mm^2$

$= 7 * 10^4 \, N/mm^2$

$$[D] = \frac{7*10^4}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} = 10^4 \begin{bmatrix} 7.692 & 2.308 & 0 \\ 2.308 & 7.692 & 0 \\ 0 & 0 & 2.692 \end{bmatrix}$$

We have,

$$[B] = \frac{1}{2\Delta}\begin{bmatrix} y_{23} & y_{31} & y_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{32} & x_{13} & x_{21} \\ x_{32} & x_{13} & x_{21} & y_{23} & y_{31} & y_{12} \end{bmatrix}$$

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 30 & 20 \\ 1 & 0 & 20 \end{vmatrix} = 300 \text{ mm}$$

$y_{23} = y_2 - y_3 = 20 - 20 = 0$

$y_{31} = y_3 - y_1 = 20 - 0 = 20$

$y_{12} = y_1 - y_2 = 0 - 20 = -20$

$x_{32} = x_3 - x_2 = 0 - 30 = -30$

$x_{13} = x_1 - x_3 = 0 - 0 = 0$

$x_{21} = x_2 - x_1 = 30 - 0 = 30$

$$[B] = \frac{1}{2*300}\begin{bmatrix} 0 & 20 & -20 & 0 & 0 & 0 \\ 0 & 0 & 0 & -30 & 0 & 30 \\ -30 & 0 & 30 & 0 & 20 & -20 \end{bmatrix}$$

$$[K] = t\Delta[B]^T[D][B]$$

$$= 10*300*\frac{1}{600}\begin{bmatrix} 0 & 0 & -30 \\ 20 & 0 & 0 \\ -20 & 0 & 30 \\ 0 & -30 & 0 \\ 0 & 0 & 20 \\ 0 & 30 & -20 \end{bmatrix}_{6*3} *10^4\begin{bmatrix} 7.692 & 2.308 & 0 \\ 2.308 & 7.692 & 0 \\ 0 & 0 & 2.692 \end{bmatrix}_{3*3}[B]$$

$$= 5*10^4\begin{bmatrix} 0 & 0 & -80.76 \\ 153.84 & 46.16 & 0 \\ -153.84 & -46.16 & 80.76 \\ -69.24 & -230.76 & 0 \\ 0 & 0 & 53.84 \\ 69.24 & 230.76 & -53.84 \end{bmatrix} * \cancel{\frac{1}{2000}}[B]$$

$$\begin{bmatrix} -153.84 & -46.16 & 80.76 \\ -69.24 & -230.76 & 0 \\ 0 & 0 & 53.84 \\ 69.24 & 230.76 & -53.84 \end{bmatrix}_{6*3} * \frac{}{2*300} \begin{bmatrix} 0 & 0 & 0 & -30 & 0 & 30 \\ -30 & 0 & 30 & 0 & 20 & -20 \end{bmatrix}_{3*6}$$

$$= \frac{250}{3} \begin{bmatrix} 2422.8 & 0 & -2422.8 & 0 & -1615.2 & 1615.2 \\ 0 & 3076.8 & -3076.8 & -1384.8 & 0 & 1384.8 \\ -2422.8 & -3076.8 & 5499.6 & 1384.8 & 1615.2 & -3000 \\ 0 & -1384.8 & 1384.8 & 6922.8 & 0 & -6922.8 \\ -1615.2 & 0 & 1615.2 & 0 & 1076.8 & -1076.8 \\ 1615.2 & 1384.8 & -3000 & -6922.8 & -1076.8 & 7999.6 \end{bmatrix} -$$

For this matrix [B], Global Dof numbering and Global load vector should be arranged in this order. since derivation of [B] comes from following arrangement.

let R_1 and R_2 be the reaction at ~~total~~ 1, in horizontal and vertical direction and R_5 and R_6 for node 3.

lobal load Vector, Global load vector

Here, no body force, no traction force.

Global Dof nubering

$$\} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{1y} \\ F_{2y} \\ F_{3y} \end{Bmatrix} = \begin{Bmatrix} 0+R_1 \\ 50 \\ 0+R_5 \\ 0+R_2 \\ -100 \\ 0+R_6 \end{Bmatrix} \begin{matrix} 1 \\ 3 \\ 5 \\ 2 \\ 4 \\ 6 \end{matrix}$$

Now, Element eq^n

$$[K]\{u\} = \{F\}$$

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 3 & 5 & 2 & 4 & 6 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \\ 2 \\ 4 \\ 6 \end{array} & \frac{250}{3}\begin{bmatrix} 2422.8 & 0 & -2422.8 & 0 & -1615.2 & 1615.2 \\ 0 & 3076.8 & -3076.8 & -1384.8 & 0 & 1384.8 \\ -2422.8 & -3076.8 & 5499.6 & 1354.8 & 1615.2 & -3000 \\ 0 & -1384.8 & 1384.8 & 6922.8 & 0 & -6922.8 \\ -1615.2 & 0 & 1615.2 & 0 & 1076.8 & -1076.8 \\ 1615.2 & 1384.8 & -3000 & -6922.8 & -1076.8 & 7999.6 \end{bmatrix} \end{array} \begin{Bmatrix} u_1 \\ u_3 \\ u_5 \\ u_2 \\ u_4 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 50 \\ R_5 \\ R_2 \\ -100 \\ R_6 \end{Bmatrix}$$

Applying boundary condition,

$$u_1 = u_2 = u_5 = u_6 = 0$$

The reduced form,

$$\frac{250}{3}\begin{bmatrix} 3076.8 & 0 \\ 0 & 1076.8 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 50 \\ -100 \end{Bmatrix}$$

$$\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 1.95 * 10^{-4} \\ -1.114 * 10^{-3} \end{Bmatrix} \text{mm}$$ Ans

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$$\begin{bmatrix} 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$= \frac{1}{2*300}\begin{bmatrix} 0 & 0 & 20 & 0 & -20 & 0 \\ 0 & -30 & 0 & 0 & 0 & 30 \\ -30 & 0 & 0 & 20 & 30 & -20 \end{bmatrix}$$

$$= t\Delta[B]^T[D][B]$$

$$= 10*300*\frac{1}{600}\begin{bmatrix} 0 & 0 & -30 \\ 0 & -30 & 0 \\ 20 & 0 & 0 \\ 0 & 0 & 20 \\ -20 & 0 & 30 \\ 0 & 30 & -20 \end{bmatrix} *10^4 \begin{bmatrix} 7.692 & 2.308 & 0 \\ 2.308 & 7.692 & 0 \\ 0 & 0 & 2.692 \end{bmatrix} [B]$$

$$5*10^4\begin{bmatrix} 0 & 0 & -80.76 \\ -69.24 & -230.76 & 0 \\ 153.84 & 46.16 & 0 \\ 0 & 0 & 53.84 \\ -153.84 & -46.16 & 80.76 \\ 69.24 & 230.76 & -53.84 \end{bmatrix} * \frac{1}{600}\begin{bmatrix} 0 & 0 & 20 & 0 & -20 & 0 \\ 0 & -30 & 0 & 0 & 0 & 30 \\ -30 & 0 & 0 & 20 & 30 & -20 \end{bmatrix}$$

$$\begin{bmatrix} 2422.8 & 0 & 0 & -1615.2 & -2422.8 & 1615.2 \\ 0 & 6922.8 & -1384.8 & 0 & 1384.8 & -6922.8 \\ 0 & -1384.8 & 3076.8 & 0 & -3076.8 & 1384.8 \\ -1615.2 & 0 & 0 & 1076.8 & 1615.2 & -1076.8 \\ -2422.8 & 1384.8 & -3076.8 & 1615.2 & 5499.6 & -3000 \\ 1615.2 & -69288 & 1384.8 & -1076.8 & 3000 & 7999.6 \end{bmatrix}$$

there is only point load.

For this matrix [B], Global load vector should be arranged in this order.

let R_1 and R_2 be the horizontal and vertical Reaction at node 1 and R_5 and R_6 be the same thing at node 2.

then, Global load vector Global Dof

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = \begin{Bmatrix} 0+R_1 \\ 0+R_2 \\ 50 \\ -100 \\ 0+R_5 \\ 0+R_6 \end{Bmatrix} \begin{matrix} \downarrow \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

Now element eqn

$$[K]\{u\} = \{F\}$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \frac{250}{3} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 2422.8 & 0 & 0 & -1615.2 & -2422.8 & 1615.2 \\ 0 & 6922.8 & -1384.8 & 0 & 1384.8 & -6922.8 \\ 0 & -1384.8 & 3076.8 & 0 & -3076.8 & 1384.8 \\ -1615.2 & 0 & 0 & 1076.8 & 1615.2 & -1076.8 \\ -2422.8 & 1384.8 & -3076.8 & 1615.2 & 5499.6 & -3000 \\ -1615.2 & -6922.8 & 1384.8 & -1076.8 & -3000 & 7999.6 \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ 50 \\ -100 \\ R_5 \\ R_6 \end{Bmatrix}$$

Applying boundary condition,

$u_1 = u_2 = u_5 = u_6 = 0$ on above eqn

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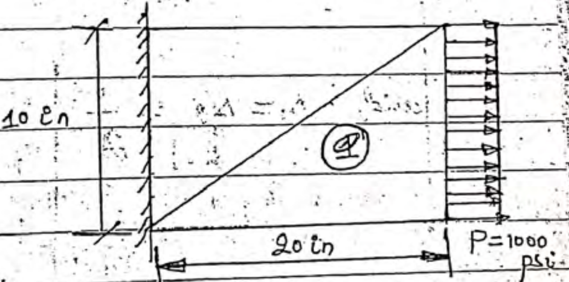
$$\frac{250}{3}\begin{bmatrix}3076.8 & 0\\ 0 & 1076.8\end{bmatrix}\begin{Bmatrix}u_3\\ u_4\end{Bmatrix}=\begin{Bmatrix}50\\ -100\end{Bmatrix}$$

$$\therefore\begin{Bmatrix}u_3\\ u_4\end{Bmatrix}=\begin{Bmatrix}1.95*10^{-4}\\ -1.114*10^{-3}\end{Bmatrix}\text{mm}$$

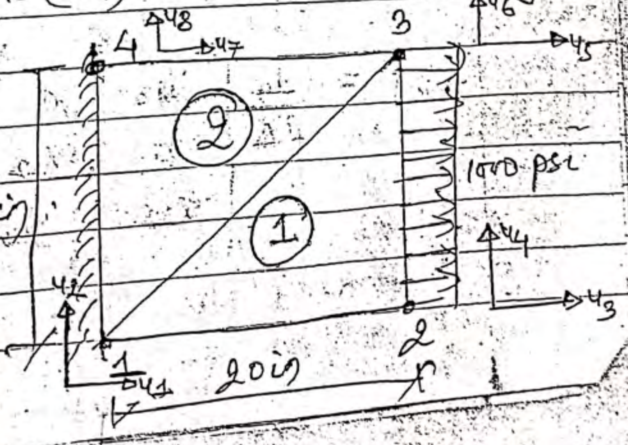
Problem No.4

The thin plate is subjected to a tensile stress p = 1000 psi. The plate thickness is 0.5 in and the ~~order~~ other dimensions are shown in figure. The poisson ratio $\nu = 0.3$ and the modulus of elasticity ~~$\nu = 0.3$~~ $E = 30*10^6$ psi. Determine the nodal displacements and the element stresses. $t = 0.5$ in



Sol^n

The given fig is divided into two constant strain triangle (CST) and node numbering is as shown in fig.



$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix} = \frac{30\times10^6}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}.$$

$$= 29.97\times10^6 \begin{bmatrix} 1.1 & 0.33 & 0 \\ 0.33 & 1.1 & 0 \\ 0 & 0 & 0.385 \end{bmatrix}$$

Element connectivity

Element No.	Nodes 1	2	3
1	1	2	3
2	1	3	4

Coordinates of nodes

coordinate	node 1	node 2	node 3	node 4
(x, y)	(0,0)	(20,0)	(20,10)	(0,10)

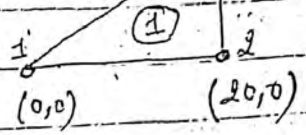
and, $\Delta_1 = \Delta_2 = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 20 & 0 \\ 1 & 20 & 10 \end{vmatrix}$

$= 100 \text{ in}^2$

We have;

$$[B] = \frac{1}{2\Delta}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \end{bmatrix}$$

$\boxed{B^1}$



$$[B^1] = \frac{1}{2*100}\begin{bmatrix}-10 & 0 & 10 & 0 & 0 & 0\\ 0 & 0 & 0 & -20 & 0 & 20\\ 0 & -10 & -20 & 10 & 20 & 0\end{bmatrix}$$

$$[K^1] = t\Delta[B]^T[D][B]$$

$$= (0.5*100)*\frac{1}{200}\begin{bmatrix}-10 & 0 & 0\\ 0 & 0 & -10\\ 10 & 0 & -20\\ 0 & -20 & 10\\ 0 & 0 & 20\\ 0 & 20 & 0\end{bmatrix} * 29.97*10^6\begin{bmatrix}1.1 & 0.33 & 0\\ 0.33 & 1.1 & 0\\ 0 & 0 & 0.385\end{bmatrix} * [B]$$

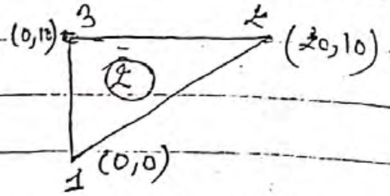
$$= 7492500\begin{bmatrix}-11 & -3.3 & 0\\ 0 & 0 & -3.85\\ 11 & 3.3 & -7.7\\ -6.6 & -22 & 3.85\\ 0 & 0 & 7.7\\ 6.6 & -22 & 0\end{bmatrix} * \frac{1}{200}\begin{bmatrix}-10 & 0 & 10 & 0 & 0 & 0\\ 0 & 0 & 0 & -20 & 0 & 20\\ 0 & -10 & -20 & 10 & 20 & 0\end{bmatrix}$$

Global Dof

$$= 37463\begin{array}{cccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & \\ 110 & 0 & -110 & -66 & 0 & -66 & 1\\ 0 & 38.5 & 77 & -38.5 & -77 & 0 & 2\\ -110 & 77 & 264 & -148 & -154 & 66 & 3\\ -66 & -38.5 & -143 & 478.5 & 77 & -440 & 4\\ 0 & -77 & -154 & 77 & 154 & 0 & 5\\ -66 & 0 & 66 & -440 & 0 & 440 & 6\end{array}$$

$\Delta = 100\ in^2$

$$[B] = \frac{1}{2*100}\begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -20 & 0 & 0 & 0 & 20 \\ -20 & 0 & 0 & 10 & 20 & -10 \end{bmatrix}$$



$$[K^2] = t\Delta\,[B^2]^T [D][B^2]$$

$$= (0.5*100)*\frac{1}{200}\begin{bmatrix} 0 & 0 & -20 \\ 0 & -20 & 0 \\ 10 & 0 & 0 \\ 0 & 0 & 10 \\ -10 & 0 & 20 \\ 0 & 20 & -10 \end{bmatrix} *29.97*10^6 \begin{bmatrix} 1.1 & 0.33 & 0 \\ 0.33 & 1.1 & 0 \\ 0 & 0 & 0.385 \end{bmatrix}[B]$$

$$= 7492500\begin{bmatrix} 0 & 0 & -7.7 \\ -6.6 & -22 & 0 \\ 110 & 3.3 & 0 \\ 0 & 0 & 3.85 \\ -11 & -3.3 & 7.7 \\ 6.6 & 22 & -3.85 \end{bmatrix} * \frac{1}{200}\begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -20 & 0 & 0 & 0 & 20 \\ -20 & 0 & 0 & 10 & 20 & -10 \end{bmatrix}$$

$$= 37463 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 154 & 0 & 0 & -77 & -154 & 77 \\ 0 & 440 & -66 & 0 & 66 & -440 \\ 0 & -66 & 110 & 0 & -110 & -66 \\ -77 & 0 & 0 & 38.5 & 77 & -38.5 \\ -154 & 66 & -110 & 77 & 264 & -143 \\ 77 & -440 & -66 & -38.5 & -143 & 478.5 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

$$= 37463 \begin{array}{|cccccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ \hline 264 & 0 & -110 & 66 & 0 & -143 & -154 & 77 & 1 \\ 0 & 478.5 & 77 & -38.5 & -143 & 0 & 66 & -440 & 2 \\ -110 & 77 & 264 & -143 & -154 & 66 & 0 & 0 & 3 \\ 66 & -38.5 & -143 & 478.5 & 77 & -440 & 0 & 0 & 4 \\ 0 & -143 & -154 & 77 & 264 & 0 & -110 & 66 & 5 \\ -143 & 0 & 66 & -440 & 0 & 478.5 & 77 & -38.5 & 6 \\ -154 & 66 & -110 & 77 & \overset{-110}{4} & \overset{77}{\cancel{-143}} & 264 & -143 & 7 \\ 77 & -440 & 0 & 0 & 66 & -38.5 & -143 & 478.5 & 8 \end{array}$$

lobal force vector

, edge 2-3 is subjected to traction force, convert it into equivalent point load at node 2-3.

have already derived,

$$] = \frac{t l_{2-3}}{6} \begin{bmatrix} 2T_{x2} + T_{x3} \\ 2T_{y2} + T_{y3} \\ T_{x2} + 2T_{y3} \\ T_{y2} + 2T_{y3} \end{bmatrix}$$

Here, $P_2 = 1000$ psi

$P_3 = 1000$ psi

$T_{x2} = 1000$ psi

$T_{y2} = 0$

$T_{x3} = 1000$ psi

$T_{y3} = 0$

-3

$$= \frac{0.5 * 10}{6} \left\{ \begin{matrix} 2*1000 + 1000 \\ 0 + 0 \\ 1000 + 2*1000 \\ 0 + 0 \end{matrix} \right\} = \left\{ \begin{matrix} 2500 \\ 0 \\ 2500 \\ 0 \end{matrix} \right\} = \left\{ \begin{matrix} F_{2x} \\ F_{2y} \\ f_{3x} \\ F_{3y} \end{matrix} \right\}$$

at node 4.

Now Global load vector;

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{Bmatrix} = \begin{array}{c|c} & \text{Global Dof} \\ \begin{Bmatrix} 0+R_1 \\ 0+R_2 \\ 2500 \\ 0 \\ 2500 \\ 0 \\ 0+R_7 \\ 0+R_8 \end{Bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} \end{array}$$

Now, element eq^n.

$$[K]\{u\} = \{F\}$$

$$37463 \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{bmatrix} 264 & 0 & -110 & 66 & 0 & -143 & -154 & 77 \\ 0 & 478.5 & 77 & -38.5 & -143 & 0 & 66 & -440 \\ -110 & 77 & 264 & -143 & -154 & 66 & 0 & 0 \\ 66 & -38.5 & -143 & 478.5 & 77 & -440 & 0 & 0 \\ 0 & -143 & -154 & 77 & 264 & 0 & -110 & 66 \\ -143 & 0 & 66 & -440 & 0 & 478.5 & 77 & -38.5 \\ -154 & 66 & 0 & 0 & -110 & 77 & 264 & -143 \\ 77 & 440 & 0 & 0 & 66 & -38.5 & -143 & 478.5 \end{bmatrix} \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ R_2 \\ 2500 \\ 0 \\ 2500 \\ 0 \\ R_7 \\ R_8 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix}$$

Now, applying boundary condition, $u_1 = u_2 = u_7 = u_8$, we get following reduced form

$$37463 \begin{bmatrix} 264 & -143 & -154 & 66 \\ -143 & 478.5 & 77 & -440 \\ -154 & 77 & 264 & 0 \\ 66 & -440 & 0 & 478.5 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 2500 \\ 0 \\ 2500 \\ 0 \end{Bmatrix}$$

$544_3 + 77u_4 + 264u_5 + 0*u_6 = 0.0067$ —⟨iii⟩

$-664_3 - 440u_4 + 0*4_5 + 478.54_6 = 0$ —⟨iv⟩

From ⟨iii⟩

$$u_5 = \frac{0.0067 + 154u_3 - 77u_4}{264} \quad \text{—⟨v⟩}$$

From (iv)

$$u_6 = \frac{-66u_3 + 440u_4}{478.5} \quad \text{—⟨vi⟩}$$

Substituting this values in eqn ⓘ & eqn ⓘⓘ, we get

$$4u_3 - 143u_4 - \left(\frac{0.0067 + 154u_3 - 77u_4}{264}\right) \cancel{+}154 + 66\left(\frac{-66u_3 + 440u_4}{478.5}\right) = 0.0667$$

$$4u_3 - 143u_4 - 0.039 - 89.83u_3 + 44.92u_4 \cancel{0} - 9.1u_3 + 60.69u_4 = 0.0667$$

$$165.07u_3 - 37.39u_4 = 0.1057 \quad \text{—⟨vii⟩}$$

eqn ⓘⓘ becomes

$$43u_3 + 478.5u_4 + 77\left(\frac{0.067 + 154u_3 - 77u_4}{264}\right) - 440\left(\frac{-66u_3 + 440u_4}{478.5}\right) = 0$$

$$43u_3 + 478.5u_4 + 0.019 + 44.92u_3 - 22.46u_4 + 60.69u_3 - 404.60u_4 = 0$$

$$-37.39u_3 + 51.44u_4 = -0.019 \quad \text{—⟨viii⟩}$$

solving (vii) and (viii)

$u_3 = 6.66 * 10^{-4}$ in

$u_4 = 1.15 * 10^{-4}$

putting this value in eqn (v) & (vi)

264

$= 6.08*10^{-4}$ in

$$u_6 = \frac{-66*6.66*10^{-4} + 440*1.015*10^{-4}}{4785}$$

$= 1.389*10^{-5}$ in

a) Nodal displacement,

$\{u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6 \quad u_7 \quad u_8\}$

$= \{0 \quad 0 \quad 6.66*10^{-4} \quad 1.015*10^{-4} \quad 6.08*10^{-4} \quad 1.389*10^{-5} \quad 0 \quad 0\}$

b) <u>Reactions</u>

From element equation (above)

~~37463 264*u_1~~ +

$37463(-110*u_3 + 66u_4 + 0*u_5 - 143*u_6) = R_1$

$37463(-110*6.66*10^{-4} + 66*1.015*10^{-4} - 143*1.389*10^{-5}) = R_1$

$R_1 = -2534.60$ lb

$37463(77u_3 - 38.5u_4 - 143u_5 + 0*u_6) = R_2$

$37463(77*6.6*10^{-4} - 38.5*1.015*10^{-4} - 143*6.08*10^{-4}) = R_2$

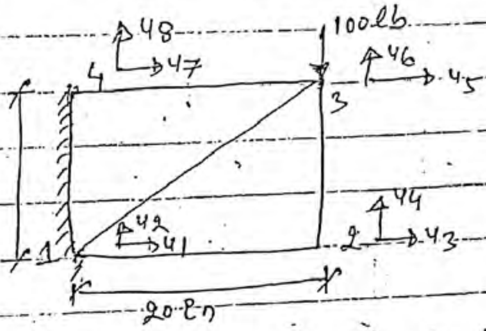
$R_2 = -1501.87$ lb

$37463(0*u_3 + 0*u_4 - 110*u_5 + 77u_6) = R_7$

$37463(-110*6.08*10^{-4} + 77*1.389*10^{-5}) = R_7$

$R_7 = -2465.46$ lb

results with values obtained from elementary beam theory



Global load vector

$$\{F\} = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} R_1 & R_2 & 0 & 0 & 0 & -100 & R_7 & R_8 \end{bmatrix}^T \end{array}$$

Now,

$[K]\{u\} = \{F\}$

$$37463 \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 264 & 0 & -110 & 66 & 0 & -143 & -154 & 77 \\ 0 & 478.5 & 77 & -38.5 & -143 & 0 & 66 & -440 \\ -110 & 77 & 264 & -143 & -154 & 66 & 0 & 0 \\ 66 & -38.5 & -143 & 478.5 & 77 & -440 & 0 & 0 \\ 0 & -143 & -154 & 77 & 264 & 0 & -110 & 66 \\ -143 & 0 & 66 & -440 & 0 & 478.5 & 77 & -38.5 \\ -154 & 66 & 0 & 0 & -110 & 77 & 264 & -143 \\ 77 & -440 & 0 & 0 & 66 & -38.5 & -143 & 478.5 \end{bmatrix} \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} = \begin{array}{c|c} \begin{Bmatrix} R_1 \\ R_2 \\ 0 \\ 0 \\ 0 \\ -100 \\ R_7 \\ R_8 \end{Bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \end{array}$$

Applying boundary condition

$u_1 = u_2 = u_7 = u_8 = 0$

$$37463 \begin{bmatrix} 264 & -143 & -154 & 66 \\ -143 & 478.5 & 77 & -440 \\ -154 & 77 & 264 & 0 \\ 66 & -440 & 0 & 478.5 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -100 \end{Bmatrix}$$

Solve find, u_3, u_4, u_5 & u_6, Find also deflection at the free end of cantilever beam from strength of material and compare it with above obtained results.

$R_8 = 1483.28$ lb

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_7 \\ R_8 \end{Bmatrix} = \begin{Bmatrix} -2534.60 \\ -1501.87 \\ -2465.46 \\ 1483.28 \end{Bmatrix} \text{ lb.}$$

<u>Check out equilibrium</u>

$$\Sigma F_x = R_1 + R_7 + 2500 + 2500$$
$$= -2534.60 - 2465.46 + 2500 + 2500$$
$$= -0.06 \approx 0 \quad \text{(OK)}$$

$$\Sigma F_y = R_2 + R_8 = -1501.87 + 1483.28$$
$$= \cancel{0} -18.59 \neq 0$$ This error may be due to finite element.

c) Strain

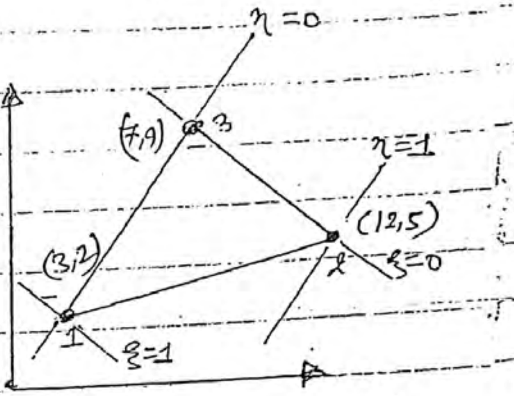
<u>For element 1</u> $\{\varepsilon^1\} = [B^1]\{u^1\} = [B^1]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$

For element 2 $\{\varepsilon^2\} = [B^2]\begin{Bmatrix} u_1 \\ u_2 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix}$

d) Stress

$\sigma^1 = E\{\varepsilon^1\}$ $\quad \sigma^2 = E\{\varepsilon^2\}$

find the area of triangle.



We have,

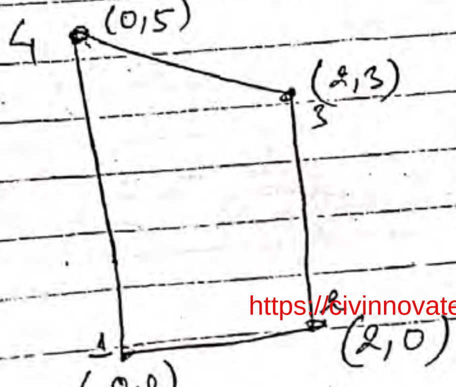
$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}$$

$$= \begin{bmatrix} 3-7 & 2-9 \\ 12-7 & 5-9 \end{bmatrix} = \begin{bmatrix} -4 & -7 \\ 5 & -4 \end{bmatrix}$$

$$\text{Area of triangle } (A) = \frac{1}{2} \left| \det[J] \right| = \frac{1}{2} \begin{vmatrix} -4 & -7 \\ 5 & -4 \end{vmatrix}$$

$$= \frac{1}{2} \times 51 = 25.5 \text{ square unit}$$

lem 7 Compute the Jacobian matrix for the following quadrilateral.



$$[J] = \begin{bmatrix} \frac{dx}{d\xi} & \frac{dy}{d\xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

We have, $N_1 = \frac{1}{4}(1-\xi)(1-\eta)$

$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$

$N_3 = 1/4\,(1+\xi)(1+\eta)$

$N_4 = 1/4\,(1-\xi)(1+\eta)$

$\frac{\partial N_1}{\partial \xi} = -\frac{1}{4}(1-\eta)$, $\frac{dN_2}{\partial \xi} = \frac{1}{4}(1-\eta)$

$\frac{\partial N_3}{\partial \xi} = \frac{1}{4}(1+\eta)$, $\frac{\partial N_4}{\partial \xi} = -\frac{1}{4}(1+\eta)$

And

$\frac{\partial N_1}{\partial \eta} = \frac{-1}{4}(1-\xi)$, $\frac{\partial N_2}{\partial \eta} = -\frac{1}{4}(1+\xi)$

$\frac{\partial N_3}{\partial \eta} = \frac{1}{4}(1+\xi)$ $\frac{\partial N_4}{\partial \eta} = \frac{1}{4}(1-\xi)$

p.t.o.

$$\ldots \frac{1}{4}(1+\xi) \quad \frac{1}{4}(1-\xi)\Big]_{2\times4} \begin{bmatrix} \vdots & \vdots \\ 2 & 3 \\ 0 & 5 \end{bmatrix}_{4*2}$$

$$= \begin{bmatrix} \frac{1}{2}(1-\eta) + \frac{1}{2}(1+\eta) & \frac{3}{4}(1+\eta) - \frac{5}{4}(1+\eta) \\ -\frac{1}{2}(1+\xi) + \frac{1}{2}(1+\xi) & \frac{3}{4}(1+\xi) + \frac{5}{4}(1-\xi) \end{bmatrix}$$

$$[J] = \begin{bmatrix} 1 & -\frac{1}{2}(1+\eta) \\ 0 & 2 - \frac{1}{2}\xi \end{bmatrix}$$ Ans

blem 7 following Fig shows quadrilateral element in global coordinates. show that mapping correctly describes the line connecting nodes 2 and 3 and determine the (x,y) coordinate corresponding to $(\xi,\eta) = (1, 0.5)$

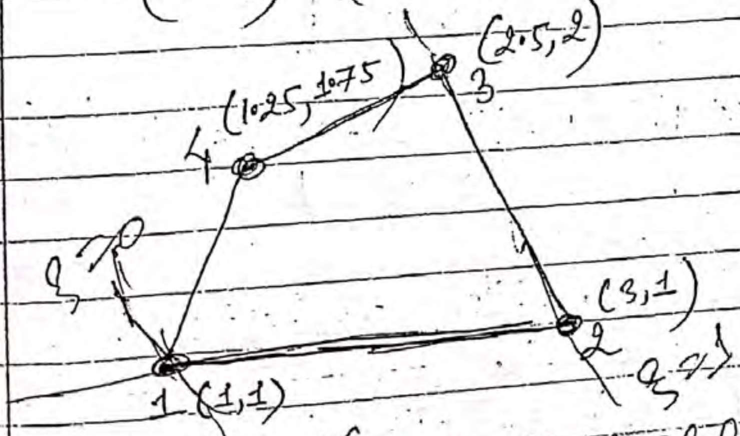


Fig:- Given Quadrilateral

through nodes 2 and 3

The eqⁿ of line passing through node 2,

$$y = mx + c$$

$$1 = 3m + c \quad \text{—} ⓘ$$

through node 3 : $2 = 2.5m + c$ —(ii)

solving ① & ②, slope $m = -2$

y intercept $6 = 7$

$\therefore$ element edge 2-3 is described by

$$y = -2x + 7$$

Now,

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$

$$= \frac{1}{4}(1-\xi)(1-\eta)(1) + \frac{1}{4}(1+\xi)(1-\eta)(3) + \frac{1}{4}(1+\xi)(1+\eta)(2.5) + \frac{1}{4}(1-\xi)(1+\xi)\times 1.25$$

$$y = \frac{1}{4}(1-\xi)(1-\eta)(1) + \frac{1}{4}(1+\xi)(1-\eta)(1) + \frac{1}{4}(1+\xi)(1+\eta)(2) + \frac{1}{4}(1-\xi)(1+\eta)1.75$$

Noting that edge 2-3 corresponds to $\xi = 1$, the above eqⁿ become

$$x = \frac{3}{2}(1-\eta) + \frac{2.5}{2}(1+\eta) = \frac{5.5}{2} - \frac{0.5}{2}\eta \quad \text{—(iii)}$$

$$y = \frac{1}{2}(1-\eta) + (1+\eta) = \frac{3}{2} + \frac{1}{2}\eta \quad \text{—(iv)}$$

eliminating η from (iii) and (iv)

$$2x + y = \frac{14}{2}$$

which is same as $y = -2x + 7$

~~Also~~, For $(\xi, \eta) = (1, 0.5)$, we obtain

$$x = \frac{5.5}{2} - \frac{0.5}{2}(0.5) = 2.625$$

$$y = \frac{3}{2} + \frac{1}{2}\eta = \frac{3}{2} + \frac{1}{2}(0.5) = 1.75$$

procedure used to solve the resulting system of algebraic equations (a solver is provided with **FEM1D**; see Appendix 1 located on the book's website at www.mhhe.com/reddy3e).

7.2.2 General Outline

A typical finite element program consists of three basic units (see Fig. 7.2.1):

1. Preprocessor
2. Processor
3. Postprocessor

In the preprocessor part of the program, the input data of the problem are read in and/or generated. This includes the geometry (e.g., length of the domain and boundary conditions), the data of the problem (e.g., coefficients in the differential equation), finite element mesh information (e.g., element type, number of elements, element length, coordinates of the nodes, and connectivity matrix), and indicators for various options (e.g., print, no print, type of field problem analyzed, static analysis, eigenvalue analysis, transient analysis, and degree of interpolation).

In the processor part, all the steps in the finite element method discussed in the preceding chapter, except for postprocessing, are performed. The major steps of the processor are:

1. Generation of the element matrices using numerical integration
2. Assembly of element equations
3. Imposition of the boundary conditions
4. Solution of the algebraic equations for the nodal values of the primary variables

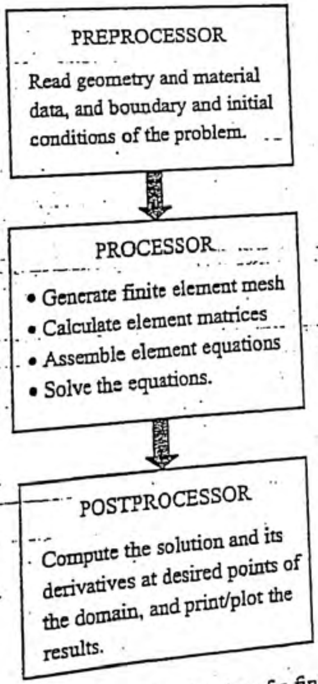


Figure 7.2.1 The three main functional units of a finite element program.

In the postprocessor part of the program, the solution is computed by interpolation at points other than nodes, secondary variables that are derivable from the solution are computed, and the output data are processed in a desired format for printout and/or plotting.

The preprocessor and postprocessors can be a few Fortran statements to read and print pertinent information, simple subroutines (e.g., subroutines to generate mesh and compute the gradient of the solution), or complex programs linked to other units via disk and tape files. The processor, where typically large amounts of computing time are spent, can consist of several subroutines, each having a special purpose (e.g., a subroutine for the calculation of element matrices, a subroutine for the imposition of boundary conditions, and a subroutine for the solution of equations). The degree of sophistication and the complexity of a finite element program depend on the general class of problems being programmed, the generality of the data in the equation, and the intended user of the program. It is always desirable to describe, through comment statements, all variables used in the computer program. A flow chart of the computer program **FEM1D** is presented in Fig. 7.2.2. The objective of each of the subroutines listed in the flow chart is described below.

5.1 Intro

FDM is a te

f(x

$f(x_2$

Real derivati

can be repre

$\frac{df}{dx} = \frac{f(x_2)-f}{x_2-x}$

Finite differ

$f(x +$

$f(x -$

Forwar

Taylor's

$f(x +$

Neglec

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